লাল-সবুজে দাগানো

TEXT BOOK

Physics

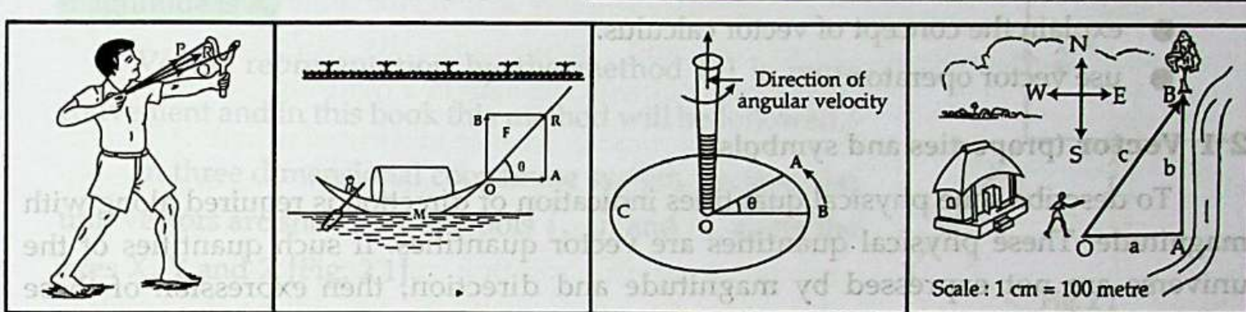
1st Paper

UNMESH

Medical & Dental Admission Care

2 VECTOR

Key Words : Vector quantity, scalar quantity, resultant and components, position vector, null vector, rectangular unit vector, resolution of vectors and components, law of triangle, law of parallelogram, scalar or dot product, vector or cross product, operator, gradient, divergence, curl.



Introduction

In order to know specifically the different aspects of science, one or the other form of measurement is needed. Physical characteristics of matter that can be measured are called quantities. As for example, each of the terms such as length, mass, time, volume, velocity, work etc is a quantity. **Any quantity in physics is called physical quantity.**

Some physical quantities can be completely expressed only by magnitude. On the other hand there are many physical quantities which need both magnitude and direction for their complete expression. Hence, according to properties or characteristics we can classify physical quantities into two categories; e.g.,

(a) Scalar quantity and

(b) Vector quantity.

(a) Scalar quantity : A physical quantity which has only magnitude and no direction is called scalar quantity or scalar. For example, length, mass, time, population, temperature, heat, electric potential, speed etc are scalar quantities.

(b) Vector quantity : A physical quantity which has both magnitude and direction is called vector quantity or vector. For example, displacement, velocity, acceleration, deacceleration or retardation, force, weight etc are vector quantities.

After studying this chapter students will be able to—

- explain properties of vector.
- explain vector representation of different physical quantities of physics.
- explain some special vector quantities.

- explain geometrical addition rule of vector quantities.
- analyse addition and subtraction of vector quantities with the help of perpendicular components.
- resolve a vector in three dimensional rectangular components.
- define scalar and vector products and their applications.
- explain the use and importance of calculus in physics.
- explain the concept of vector calculus.
- use vector operator.

2·1 Vector (properties and symbols)

To describe sone physical quantities indication of direction is required along with magnitude. These physical quantities are vector quantities. If such quantities of the universe are not expressed by magnitude and direction, then expression of these quantities remain incomplete. Vector quantities follow or obey some rules; e.g.—

1. Vector quantity has magnitude and direction.
2. Two or more homogeneous vectors can be added. Different kind of vectors cannot be added.
3. When two or more vectors are added, then the resultant vector is equal to the result of total action of the first two vectors.
4. The vector product of two vectors is a vector quantity.
5. The scalar product of two vectors is a scalar quantity.
6. The direction of a vector is indicated by the proportion of the vector quantity and its magnitude.
7. Vector quantity obeys the rules of addition and distribution.
8. Vector quantity can be divided into components.

A vector quantity is expressed in two ways by sign, e.g.—**by letter or by straight line. By letter a vector quantity can be divided in four ways, e.g.—**

(a) Vector form of a quantity is expressed by putting an arrow over a letter and its magnitude is expressed by placing two vertical lines across the letter. Normally, magnitude of the quantity is also expressed only by the letter.

∴ vector form of word A is $\vec{A}$ and its magnitude is $|\vec{A}|$ or simply A.

(b) Vector quantity is expressed by putting a line above a letter and its magnitude is expressed by two vertical lines across the letter.

∴ vector form of A is $\bar{A}$ and its magnitude is $|\vec{A}|$

(c) Vector quantity is represented by putting a line underneath the letter and its magnitude is expressed by two vertical lines across that letter. For example, vector representation of A is $\underline{A}$ and its magnitude is $|\underline{A}|$.

(d) A vector quantity is represented by a bold letter. For example vector form of A is **A** and its magnitude is A.

Vector representation by the method (a) is more convenient and in this book this method will be followed.

In three dimensional coordinate system, rectangular unit vectors are shown by symbols $\hat{i}$, $\hat{j}$ and $\hat{k}$ along the axes X, Y and Z [Fig. 2·1].

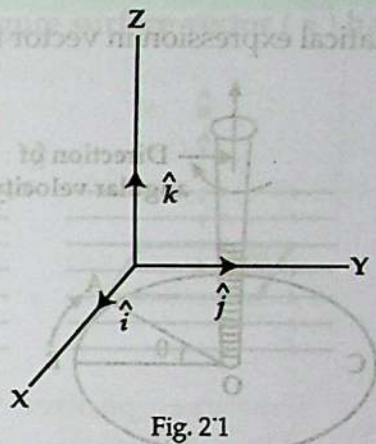


Fig. 2·1

2·2 Vector representation

Different vector quantities are mathematically expressed in vector form. As a result the magnitude and direction of the quantity becomes clear. Vector forms of some physical quantities are shown below.

Force : The external cause which changes or tends to change the state of rest or motion of a body is called force.

Force is a vector quantity. It has both direction and magnitude. If the mass of a moving body is 'm', its acceleration is $\vec{a}$ and force acting on this body is $\vec{F}$, then the vector representation is

$$\vec{F} = m\,\vec{a}$$

Rotational force or torque

Rotational force means torque. In case of rotational object equivalent quantity of force is torque.

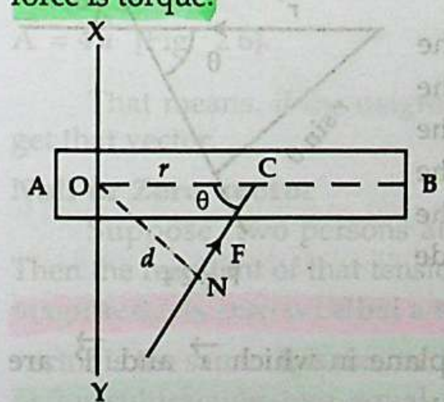


Fig. 2·2

The product of the position vector and applied force on a point of a body rotating around an axis is called the rotational force or torque.

If $\vec{r}$ is the position vector and $\vec{F}$ is the apllied force. then,

torque, $\vec{\tau} = \vec{r} \times \vec{F}$ [Fig. 2·2]

Torque is a vector quantity. The direction of τ is perpendicular to a plane that contains $\vec{r}$ and $\vec{F}$.

Angular velocity

If the time interval approaches zero, the rate of change of angular displacement with time is called instantaneous angular velocity or angular velocity ($\vec{\omega}$). Its mathematical expression in vector form is $\vec{v} = \vec{\omega} \times \vec{r}$

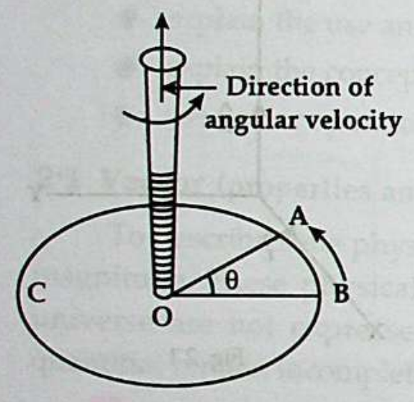


Fig. 2·3

Direction of angular velocity (Vector form)

Angular velocity is a vector quantity. It has both direction and magnitude. Its direction can be demonstrated with the help of a right-turn screw.

If a particle is rotated in a circular path ABC and by placing the top of a right-turn screw perpendicularly at the centre of that plane of the circle is allowed to rotate along the rotation of the particle, then the direction of rotation of the screw will be the angular velocity of the particle [Fig. 2·3].

Angular momentum

The vector product of the radius vector and the linear momentum of a revolving particle is called angular momentum.

Explanation : Suppose $\vec{r}$ = radius vector of a particle rotating with respect to its centre of rotation and $\vec{P}$ = linear momentum of the body. Hence, according to the definition the angular momentum is

$$\vec{L} = \vec{r} \times \vec{P}$$

It is a vector quantity.

Magnitude and direction : The magnitude of angular momentum is

$$L = rP \sin \theta$$

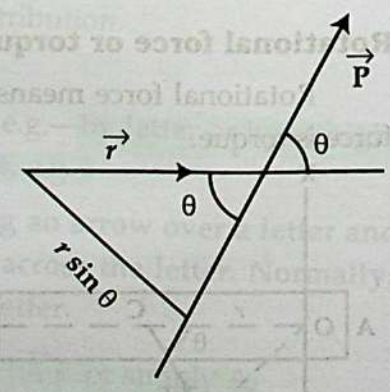


Fig. 2·4

Here θ is the angle between $\vec{r}$ and $\vec{P}$ [Fig. 2·4]. The perpen-dicular distance of the line of action of the momentum from the centre of rotation is $r \sin \theta$. So, the product of linear momentum of any particle and the perpendicular distance of the line of action of the momentum from the centre of rotation gives the magnitude of the angular momentum.

The direction of $\vec{L}$ will be perpendicular to the plane in which $\vec{r}$ and $\vec{P}$ are located. The direction $\vec{L}$ will be determined by cross product.

Surface

The direction represented by the normal drawn on a plane or surface is the vector of that surface. In this case the plane is the surface. The normal drawn on a surface is called the surface vector of that surface. In the following figure surface vector ($\hat{n}$) has been shown by hypen line. Here the coil is the surface.

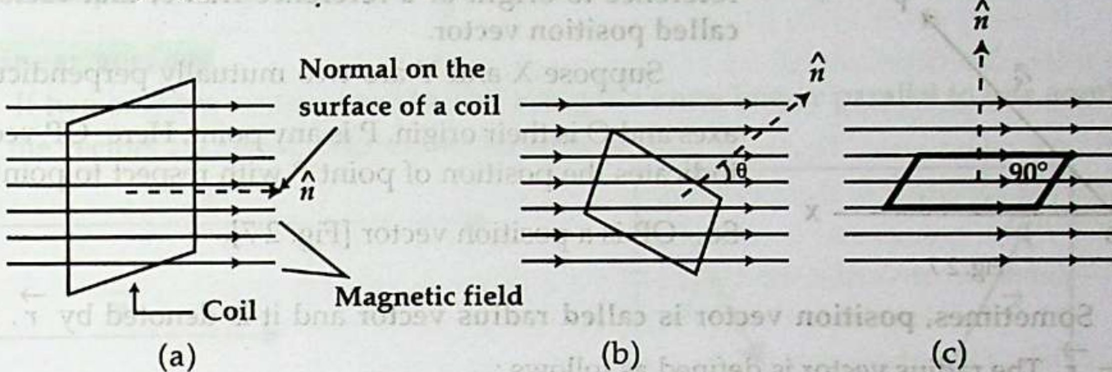


Fig. 2·5

2·3 Special vectors

Unit vector

When a vector of non-zero magnitude is divided by its magnitude it gives a unit vector along the direction or parallel to the direction of that vector. That means, **a vector whose magnitude is a unit is called unit vector.**

In order to describe a unit vector normally a hat sign (^) is put on the small letter. For example, $\hat{i}$, $\hat{a}$, $\hat{n}$ etc are used for denoting unit vector.

Let $\vec{A}$ be a vector whose magnitude, $A \neq 0$

$\therefore$ $\frac{\vec{A}}{A}$ = **unit vector along the direction of** $\vec{A} = \hat{a}$ (suppose).

So, if the magnitude of a vector $\vec{A}$ is A = 4 unit, and if $\hat{a}$ is the unit vector along the direction of A, then $\vec{A} = 4\hat{a}$ [Fig. 2·6].

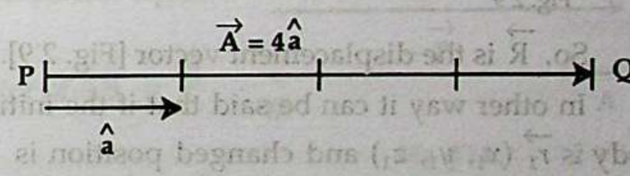


Fig. 2·6

That means, if the magnitude of a vector is multiplied by its unit vector, then we get that vector.

Null or Zero vector

Suppose, two persons at two ends of a rope are pulling it with the same force. Then the resultant of that tension will be a null vector. In other words, the vector whose magnitude is zero is called a null or zero vector. The initial and the end point of a zero vector is the same. It is denoted by $\vec{0}$ or 0. By adding a vector with its opposite vector or by subtracting two equal vectors, a null vector is obtained. It has no particular direction.

Position vector

A vector is needed in order to know the position of a point in the reference frame. The position of the body can be determined by the magnitude and direction of that vector.

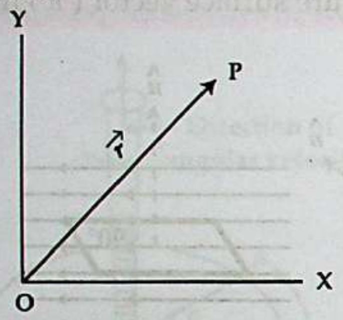


Fig. 2·7

When the position of a vector is specified with reference to origin of a reference frame, that vector is called position vector.

Suppose X and Y are two mutually perpendicular axes and O is their origin. P is any point. Here $\overrightarrow{OP}$ vector indicates the position of point P with respect to point O. So, $\overrightarrow{OP}$ is a position vector [Fig. 2·7].

Sometimes, **position vector is called radius vector** and it is denoted by $\vec{r}$. So, $OP = \vec{r}$. The radius vector is defined as follows :

Radius vector

The distance of a point from the origin is called radius vector. It is denoted by $\vec{r}$ [Fig. 2·8].

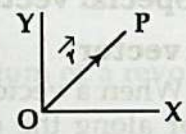


Fig. 2·8

Explanation : Here, the distance of the position of point P from the origin is called the radius vector.

Displacement vector

Distance travelled by a point along a linear or straight path is called the displacement vector. It is represented by $\vec{R}$.

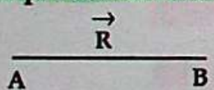


Fig. 2·9

Explanation : Suppose, the distance travelled by a point along a straight path is AB = R.

So, $\vec{R}$ is the displacement vector [Fig. 2·9].

In other way it can be said that if the initial position of a body is $\vec{r_1}$ (x_1, y_1, z_1) and changed position is $\vec{r_2}$ (x_2, y_2, z_2), then displacement vector $\Delta \vec{r} = \vec{r_2} - \vec{r_1}$ [Fig. 2·10].

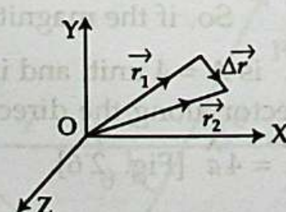


Fig. 2·10

Like vectors

If two vectors $\vec{A}$ and $\vec{B}$ of same type having unequal magnitude are parallel to each other and directed along the same direction then they are called like vectors [Fig. 2·11].

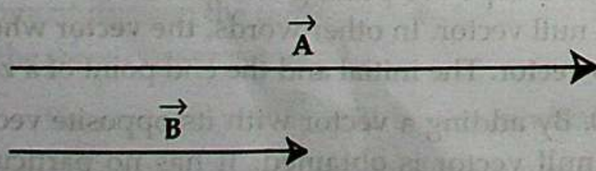


Fig. 2·11

Example : $\vec{A} = 2\vec{B}$

Reciprocal vectors

If the magnitude of one of the two parallel vectors is reciprocal of the other, then they are called reciprocal vectors.

Example : $\vec{A} = 5\,\hat{i}$ and $\vec{B} = \frac{1}{5}\,\hat{i}$, here $\vec{A}$ and $\vec{B}$ are reciprocal vectors.

Collinear vectors

If two or more vectors are directed along the same line or parallel to one another, then the vectors are called collinear vectors [Fig. 2·12].

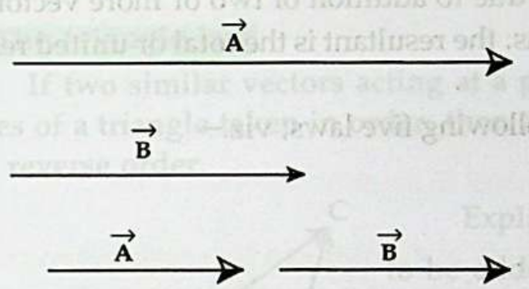


Fig. 2·12

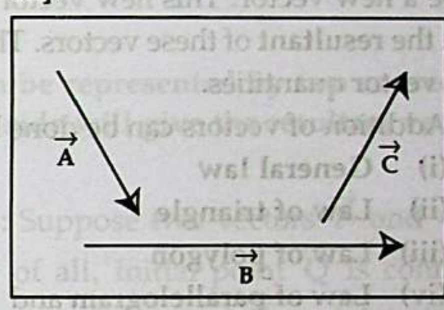


Fig. 2·13

Coplanar vectors

If two or more vectors are parallel to (or lie on) the same plane, then the vectors are called coplanar vectors [Fig. 2·13].

Negative vector and Equal vector

If the magnitudes of two similar vectors acting opposite to each other are equal, then those vectors are called opposite or negative vectors of one another. And if two similar vectors have same magnitude and are in the same direction, then they are called equal vectors.

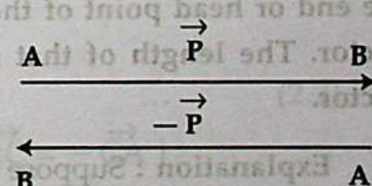


Fig. 2·14

In fig. 2·14 opposite or negative vector of $\overrightarrow{AB} = \vec{P}$ is $\overrightarrow{BA} = -\vec{P}$

here $\overrightarrow{AB} = \overrightarrow{BA}$

2·4 Rules of geometrical addition of vectors

Geometrical Method : Two similar vectors can be added or subtracted. For example, displacement can be added only with displacement. Question does not arise to add or subtract displacement with velocity. **A vector quantity has both magnitude and direction. So, addition or subtraction of vector quantities is not done by usual algebraic rule. It is the directions of vectors that create problem in this case.** For example let the velocity of the oar of a boat be 8 km/hr, and the velocity of current in the river be 6 km/hr. If the boat is allowed to move straight from one bank of the river to the opposite bank, the velocity that will act on the boat will not be equal to the algebraic addition of (8 + 6) = 14 km/hr. This will not give actual velocity—actual velocity will be totally different. Besides, the direction of the boat will be between the directions of the two velocities. For this reason, addition and subtraction of vector quantities are done following geometrical method.

In order to add two like vector quantities, the quantities are to be directed along the same direction; on the other hand for subtraction the quantities are to be directed opposite to each other. But if two or more vectors act at the same point, their addition will be a new vector. This new vector formed due to addition of two or more vectors is called the **resultant** of these vectors. That means, the resultant is the **total** or **united** result of the vector quantities.

Addition of vectors can be done by the following five laws; viz.—

(i) General law
(ii) Law of triangle
(iii) Law of polygon
(iv) Law of parallelogram and
(v) Law of components.

In this section we shall discuss the first four laws.

1. General law

Of the two vectors, the end or head point of the first vector and the initial point or tail point of the second vector are placed on the same point, then the direction of the straight line connecting the initial or tail point of the first vector and the end or head point of the second vector will give the direction of the resultant vector. The length of that straight line will give the magnitude of the resultant vector.

Explanation : Suppose two vectors $\vec{P}$ and $\vec{Q}$ are acting at the same time at the same point and we need to find out the resultant vector $\vec{R}$.

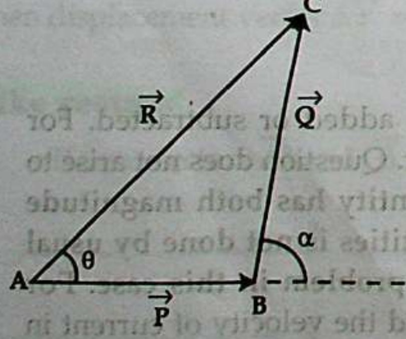


Fig. 2·15

Let the tail point of the vector $\vec{Q}$ represented by line BC be placed at the head or arrow point of the vector $\vec{P}$ represented by line AB [Fig. 2·15]. Now, let us draw the line AC connecting the tail point of $\vec{P}$ and head point of $\vec{Q}$, then the line AC will give the resultant vector $\vec{R}$. So, the addition of vectors $\vec{P}$ and $\vec{Q}$ is

$$\vec{R} = \vec{P} + \vec{Q} \quad \ldots \quad \ldots \quad \ldots \quad (2·1)$$

Similarly, more than two vectors can be added.

In fig. 2·16, three vectors $\vec{P}$, $\vec{Q}$ and $\vec{S}$ have been indicated by straight lines OA, AB, BC and by straight line OC their resultant has been indicated by $\vec{R}$.

Here, resultant, $\vec{R} = \vec{P} + \vec{Q} + \vec{S}$

Again, if $\vec{P}$, $\vec{Q}$, $\vec{R}$ act along the three arms of a triangle then, $\vec{P} + \vec{Q} + \vec{R} = 0$

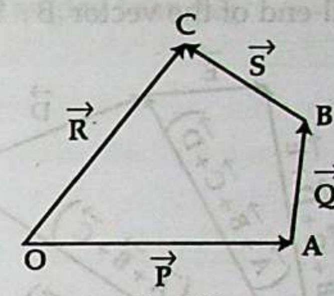


Fig. 2·16

2. The triangle law

If two similar vectors acting at a point can be represented by two consecutive sides of a triangle taken in order, then the third side will give the resultant vector in the reverse order.

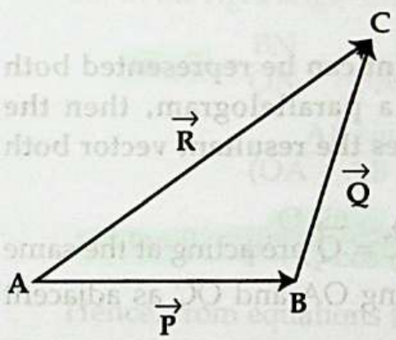


Fig. 2·17

Explanation : Suppose two vectors $\vec{P}$ and $\vec{Q}$ are to be added. First of all, initial point $\vec{Q}$ is connected with the terminal or end point of $\vec{P}$ and let the two vectors are represented in magnitude and direction by arms AB and BC. Now, let the initial point of $\vec{P}$ and the end point of $\vec{Q}$ be added and complete the triangle ABC. The arm AC represents in magnitude and direction the resultant vector $\vec{R}$ of $\vec{P}$ and $\vec{Q}$ [Fig 2·17].

That means,

$$\vec{AB} + \vec{BC} = \vec{AC}$$

or, $\vec{P} + \vec{Q} = \vec{R}$ ... ... (2.2)

Again, $\vec{AB} + \vec{BC} = \vec{AC} = -\vec{CA}$ $[\because \vec{AC} = -\vec{CA}]$

or, $\vec{AB} + \vec{BC} + \vec{CA} = 0$ ... ... (2.3)

Conclusion : If three vectors acting simultaneously at a point are represented by three sides of a triangle taken in order, then the resultant will be zero.

3. Law of polygon

This law states that if a vector polygon be drawn, placing the tail-end of each succeeding vector at the head or arrow-end of the preceding one, their resultant is drawn from the tail-end of the first to the head or arrow-end of the last.

Explanation : Let us consider vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$ [Fig. 2·18] and their resultant vector is to be found out. Now, we place on the head or arrow-end of the first vector $\vec{A}$

the tail-end of the vector $\vec{B}$. Then on the head or arrow-end of $\vec{B}$ the tail-end of $\vec{C}$ and following the same procedure other vectors are placed successively [Fig. 2·18]. Now connect the tail-end of vector $\vec{A}$ and the head or arrow-end of the last vector i,e., $\vec{E}$. Then, according to the law of polygon, this connecting vector $\vec{R}$ will indicate the magnitude and direction of the resultant of the mentioned vector quantities.

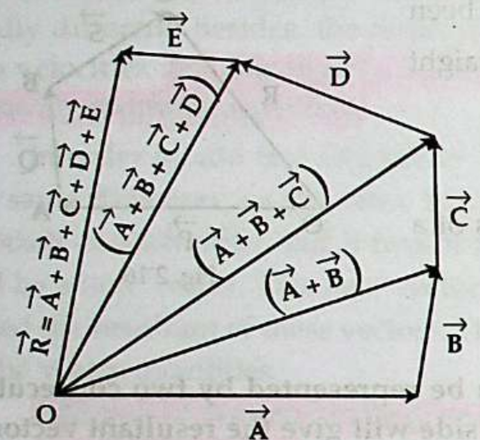


Fig. 2·18

$\therefore$ Resultant, $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$

4. The parallelogram law

If two similar vectors acting simultaneously at a point can be represented both in magnitude and direction by two adjacent sides of a parallelogram, then the diagonal from the point of intersection of these sides gives the resultant vector both in magnitude and direction.

Explanation : Suppose, two vectors $\vec{OA} = \vec{P}$ and $\vec{OC} = \vec{Q}$ are acting at the same time at point O of a particle at an angle α [Fig. 2·19]. Taking OA and OC as adjacent sides let us draw a parallelogram OABC and connect OB.

Now, according to the parallelogram law, the diagonal OB drawn from the tail-points of $\vec{P}$ and $\vec{Q}$ represents the resultant vector $\vec{R}$ i.e.,

$$\vec{OA} + \vec{OC} = \vec{OB}$$

or, $\vec{P} + \vec{Q} = \vec{R}$

Determination of magnitude of the resultant :

Suppose the magnitude of the resultant is R and $\angle AOC = \alpha$ is an acute angle [Fig. 2·19]. Now from point B let us draw normal BN on the extended line of OA which intersects the line OA at N.

Now, AB and OC are parallel

$\therefore \angle AOC = \angle BAN = \alpha$

Again, in ΔOBN, $\angle ONB = 90°$

$\therefore OB^2 = ON^2 + BN^2 = (OA + AN)^2 + BN^2$

$= OA^2 + 2OA.AN + AN^2 + BN^2$

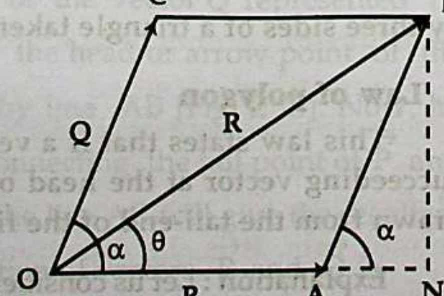


Fig. 2·19

But, $AB^2 = AN^2 + BN^2$ [$\because$ ΔBNA is a right angled triangle]

or, $OC^2 = AN^2 + BN^2$ [$\because$ AB = OC]

Now from trigonometry,

$AN = AB \cos \alpha = OC \cos \alpha$

So, $OB^2 = OA^2 + OC^2 + 2OA \,.\, AB \cos \alpha$

or, $OB^2 = OA^2 + OC^2 + 2OA \,.\, OC \cos \alpha$

or, $R^2 = P^2 + Q^2 + 2PQ \cos \alpha$

$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ \cos \alpha}$ ... ... (2·4)

Determination of direction of the resultant :

Suppose the resultant R makes an angle θ with P, i.e., ∠AOB = θ.

So, in the right angle triangle ΔOBN,

$$\tan \theta = \frac{BN}{ON} = \frac{BN}{(OA + AN)}$$

$$= \frac{AB \sin \alpha}{(OA + AB \cos \alpha)} \quad \left[\because \sin \alpha = \frac{BN}{AB} \text{ and } \cos \alpha = \frac{AN}{AB}\right]$$

$\therefore \quad \tan \theta = \dfrac{Q \sin \alpha}{P + Q \cos \alpha}$ ... ... (2·5)

Hence, from equations (2·4) and (2·5) we get R and θ respectively.

Investigation : Under what condition the magnitude of the resultant of two equal vectors can be equal to the magnitude of each of them ?

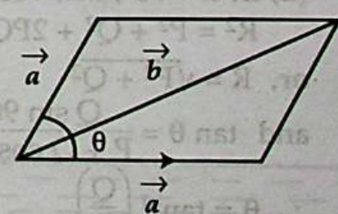
Let the magnitude of each of the vectors be a, angle between the two vectors be θ and the resultant of the two vectors be b.

$\therefore \quad b^2 = a^2 + a^2 + 2a \,.\, a \cos \theta$

or, $b^2 = 2a^2 + 2a^2 \cos \theta$

or, $b^2 = 2a^2 (1 + \cos \theta)$

$\therefore \quad b = a \sqrt{2(1 + \cos \theta)}$



Now, if $\sqrt{2(1 + \cos \theta)} = 1$, then there will be $b = a$

or, $2(1 + \cos \theta) = 1$

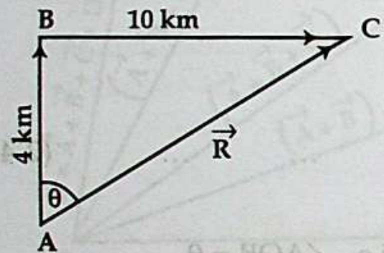
or, $\cos \theta = \frac{1}{2} - 1 = -\frac{1}{2} = \cos 120°$

or, θ = 120°

Hence, if the angle between the two vectors is 120° then the magnitude of the resultant will be equal to the magnitude of each of them.

Calculate : Calculate the resultant displacement if the displacement of a particle is 4 km and 10 km respectively along the North and the East.

Vectors $\vec{AB}$ and $\vec{BC}$ represent 4 km and 10 km displacements respectively along the North and the East. According to the law of triangle the resultant of the displacement vector $\vec{AC}$ is drawn.



If the angle between $\vec{AB}$ and $\vec{AC}$ is θ and the magnitude of the resultant vector is R, then

$$R = \sqrt{10^2 + 4^2} = \sqrt{116} = 10{\cdot}78 \text{ km}$$

$$\text{and } \tan\theta = \frac{10}{4} = 2{\cdot}5 \quad \therefore \theta = 68{\cdot}2°$$

Some special cases :

(i) If $\alpha = 0°$, i. e., vectors $\vec{P}$ and $\vec{Q}$ act in the same direction or the vectors are parallel to each other, then

$$\begin{aligned} R^2 &= P^2 + Q^2 + 2PQ\cos 0° \\ &= P^2 + Q^2 + 2PQ \quad [\because \cos 0° = 1] \\ &= (P+Q)^2 \\ \therefore R &= P + Q \end{aligned}$$

$$\text{and } \tan\theta = \frac{Q\sin 0°}{P + Q\cos 0°} = \frac{0}{P+Q} = 0 \quad [\because \sin 0° = 0]$$

$$\theta = \tan^{-1} 0 = 0°$$

$$\therefore \theta = 0°$$

So, the magnitude of the resultant of the two vectors acting in the same direction is the sum of the magnitudes of each vector and the direction of the resultant is in the direction along which the vectors act.

(ii) If, $\alpha = 90°$, i.e., vectors $\vec{P}$ and $\vec{Q}$ are perpendicular to each other, then

$$R^2 = P^2 + Q^2 + 2PQ\cos 90° = P^2 + Q^2 \quad [\because \cos 90° = 0]$$

$$\text{or, } R = \sqrt{P^2 + Q^2}$$

$$\text{and } \tan\theta = \frac{Q\sin 90°}{P + Q\cos 90°} = \frac{Q}{P}$$

$$\therefore \theta = \tan^{-1}\left(\frac{Q}{P}\right)$$

That means, if two vectors act perpendicular to each other, then their resultant will be equal to the square root of the addition of squares of the two quantities.

(iii) If $\alpha = 180°$ i.e., if the two vectors are in opposite direction to each other, then

$$\begin{aligned} R^2 &= P^2 + Q^2 + 2PQ\cos 180° \\ &= P^2 + Q^2 - 2PQ \quad [\because \cos 180° = -1] \end{aligned}$$

$$\text{or, } R^2 = (P \sim Q)^2$$

$$\therefore R = P \sim Q$$

and $\tan\theta = \dfrac{Q\sin 180°}{P + Q\cos 180°} = \dfrac{0}{P - Q} = 0$ [$\because \sin 180° = 0$]

$\therefore\ \theta = \tan^{-1} 0 = 180°$ or, $0°$

i.e., if vectors are opposite in direction to each other, then the magnitude of the resultant will be the subtraction of the two vectors and the direction of the resultant will be along the direction of the larger vector. But if the vectors are equal and opposite in direction, then the resultant will be zero.

Considering the above cases, following results are found. When the magnitudes of the two vectors are a and b and c represents their resultant, then

(i) the maximum value of the resultant of the two vectors $c = a + b$, when the two principal vectors are in the same direction.

(ii) the minimum value of the resultant of the two vectors $c = a \sim b$, when the two principal vectors are in the opposite direction.

(iii) If the magnitude of the two vectors is same and are in the opposite direction, then magnitude of their resultant becomes zero.

Work : R is the resultant vector of two vectors 3F and 3F. If the first vector is double, then the resultant also becomes double. Calculate the angle between the vectors.

Mathematical examples

1. The velocity of a rowing boat in favour of the current in a river is 18 km/hr and against the current it is 6 km/hr. In which direction the boat is to be driven in order to reach the other side of the river and what will be the velocity of the boat ?

Let the velocity of the current = u and

the velocity of rowing = v

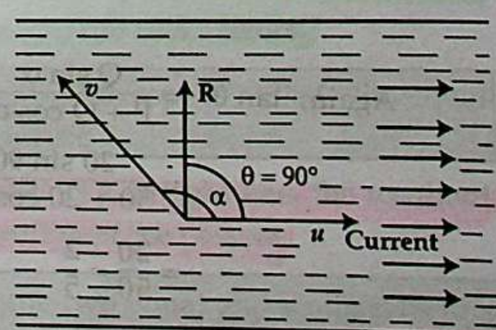
Here, $u + v = 18\ \text{kmh}^{-1}$

and $v - u = 6\ \text{kmh}^{-1}$

$\therefore$ by addition and subtraction we get,

$v = 12\ \text{kmh}^{-1}$ and

$u = 6\ \text{kmh}^{-1}$



Let us consider that the boat will reach the other side of the river if it is driven with velocity R at an angle α with the current.

Then the component of R along the current is,

$$R\cos 90° = 0 = u\cos 0° + v\cos\alpha$$

$$\therefore\ \cos\alpha = -\frac{u}{v} = -\frac{1}{2} = \cos 120°$$

or, $\alpha = 120°$

Again, the component of R perpendicular to the current is,

$$R \sin 90° = R = u \sin 0° + v \sin \alpha$$

$$\therefore \quad R = v \sin \alpha = v\sqrt{1-\cos^2 \alpha}$$

$$= 12\sqrt{1-\left(-\frac{1}{2}\right)^2}$$

$$= 6\sqrt{3} = 10{\cdot}39 \text{ kmh}^{-1}$$

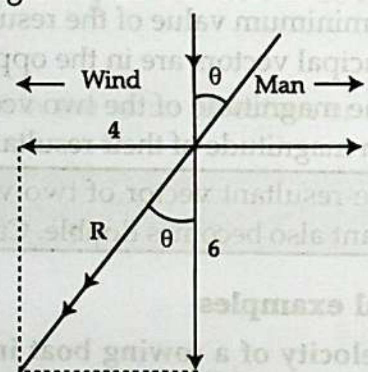
2. A man while running at velocity 4 ms⁻¹ comes across rain falling vertically with velocity 6 ms⁻¹. At what angle he will have to hold an umbrella to protect himself from rain ?

Let the resultant velocity of rain make an angle θ with the vertical direction.

$$\therefore \quad \tan\theta = \frac{4 \text{ ms}^{-1}}{6 \text{ ms}^{-1}} = 0{\cdot}666$$

$$\text{or,} \quad \tan\theta = \tan 33{\cdot}7°$$

$$\therefore \quad \theta = 33{\cdot}7°$$



So, the man must hold the umbrella at 33·7° with the vertical direction.

3. A body is pulled towards west by a froce of 50 N and towards north by a force of 20 N. Find the magnitude and direction of the resultant force.

We know,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$= \sqrt{(50)^2 + (20)^2 + 2 \times 50 \times 20 \times \cos 90°}$$

$$= \sqrt{(50)^2 + (20)^2} = \sqrt{2500 + 400}$$

$$= \sqrt{2900} \text{ N} = 53{\cdot}85 \text{ N}$$

Here,

$P = 50$ N

$Q = 20$ N

$\alpha = 90°$

$R = ?$

$\theta = ?$

$$\text{Again,} \quad \tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}$$

$$= \frac{20 \sin 90°}{50 + 20\cos 90°}$$

$$= \frac{20}{50} = \frac{2}{5}$$

$$\therefore \quad \theta = \tan^{-1}\frac{2}{5} = 21{\cdot}80°$$

Maximum and minimum value of the resultant

Suppose two vectors $\vec{P}$ and $\vec{Q}$ are acting simultaneously at a point making an angle α. Now according to parallelogram law the magnitude of the resultant,

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

From the above equation it is evident that R depends on the angle between $\vec{P}$ and $\vec{Q}$ i.e., on α.

(i) Maximum value of the resultant : R will be maximum when $\cos\alpha$ will be maximum i.e., when $\cos\alpha = 1 = \cos 0°$

or, $\alpha = 0°$

$$R_{(max.)} = \sqrt{P^2 + Q^2 + 2PQ\cos 0°}$$
$$= \sqrt{(P + Q)^2}$$
$$= (P + Q) \quad \ldots \quad \ldots \quad \ldots \quad (2{\cdot}6)$$

Thus, **when two vectors act along the same straight line then the magnitude of the resultant will be maximum. The magnitude will be the sum of the vectors. In other words, we can say that magnitude of the resultant cannot be greater than the sum of the vectors.**

(ii) Minimum value of the resultant : The magnitude of $\vec{R}$ will be minimum when $\cos\alpha$ is minimum i.e., $\cos\alpha = -1 = \cos 180°$ or, $\alpha = 180°$.

$$\therefore R_{(min.)} = \sqrt{P^2 + Q^2 + 2PQ\cos 180°}$$
$$= \sqrt{P^2 + Q^2 - 2PQ}$$
$$= \sqrt{(P \sim Q)^2} = P \sim Q \quad \ldots \quad \ldots \quad \ldots \quad (2{\cdot}7)$$

Thus, **when two vectors act along the same line but in opposite direction, the magnitude of the resultant will be minimum and the magnitude of the minimum value of the resultant will be equal to the subtraction of the two vectors.**

Work : By holding the two ends of a rope by two persons the rope is pulled with equal force. Will there be any change of the position of the persons ?

If two equal and opposite forces act along the ends of a straight line, the resultant will be zero. Hence, there will not be any change of position of the persons.

Inquisitive work : Explain whether the resultant of the two equal and similar vectors can be zero or not.

Mathematical example

1. The maximum resultant of two forces is 10 N and the minimum resultant is 4 N; if the two forces act on a particle making an angle of 90° with each other, then what is the magnitude of the resultant ?

We know,

$R_{max} = P + Q$

or, $10 = P + Q$ ... ... (i)

Again, $R_{min} = P - Q$

or, $4 = P - Q$ ... ... (ii)

Here,

$R_{max} = 10$ N

$R_{min} = 4$ N

$\alpha = 90°$

R = ?

Adding equation (i) and equation (ii) we get,

$2P = 14$ N $\quad \therefore \quad P = \frac{14}{2}$ N $= 7$ N

Again, subtracting equation (i) from equation (ii) we get,

$2Q = 6$ N $\quad \therefore \quad Q = 3$ N

Again, $R^2 = P^2 + Q^2 + 2PQ \cos 90° = P^2 + Q^2$ [$\because \cos 90° = 0$]

$= (7)^2 + (3)^2 = 49 + 9 = 58$

$\therefore \quad R = \sqrt{58}$ N

2·5 Some laws of vector addition

(i) Commutative law : $\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$

Proof : Suppose $\vec{P}$ and $\vec{Q}$ are two vectors and $\vec{R}$ is their resultant [Fig. 2·20].

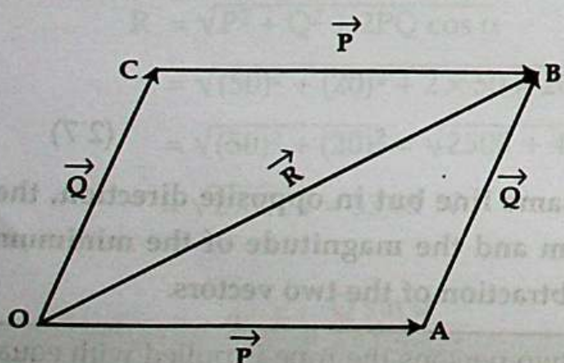


Fig. 2·20

Now, from the law of triangle, in ΔOAB triangle, $\vec{R} = \vec{P} + \vec{Q}$

i.e., $\vec{OB} = \vec{OA} + \vec{AB}$

Let us draw the parallelogram OABC in which OC and CB are respectively AB and OA. In ΔOCB triangle,

$\vec{OB} = \vec{OC} + \vec{CB}$

(According to law of triangle)

$\therefore \quad \vec{OA} + \vec{OB} = \vec{OC} + \vec{CB}$

i.e., $\vec{P} + \vec{Q} = \vec{Q} + \vec{P}$ ... ... (2.8)

This is the commutative law.

Scalar quantities also follow commutative law.

(ii) Associative law : $(\vec{P}+\vec{Q})+\vec{R}=\vec{P}+(\vec{Q}+\vec{R})$

Suppose P, Q and R are three vectors [Fig. 2·21], which have been represented by $\vec{AB}$, $\vec{BC}$ and $\vec{CD}$ lines respectively.

Now, join AC, BD and AD. Using the law of triangle

in ΔABC triangle, we get, $\vec{AC} = \vec{AB} + \vec{BC} = \vec{P} + \vec{Q}$

In ΔACD triangle, we get, $\vec{AD} = \vec{AC} + \vec{CD}$

$$= \left(\vec{P} + \vec{Q}\right) + \vec{R} \quad \ldots \quad (2.9)$$

Again, in ΔBCD triangle,

$\vec{BD} = \vec{BC} + \vec{CD} = \vec{Q} + \vec{R}$

and in ΔABD triangle,

$$\vec{AD} = \vec{AB} + \vec{BD} = \vec{P} + \left(\vec{Q} + \vec{R}\right) \ldots \quad (2.10)$$

∴ From equations (2.9) and (2.10), we get,

$$\left(\vec{P} + \vec{Q}\right) + \vec{R} = \vec{P} + \left(\vec{Q} + \vec{R}\right)$$

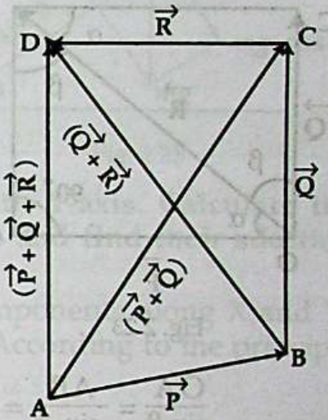


Fig. 2·21

This is the Associative law of vector addition. It means that vector addition obeys associative law.

(iii) Distributive law : $m(\vec{P}+\vec{Q}) = m\vec{P} + m\vec{Q}$

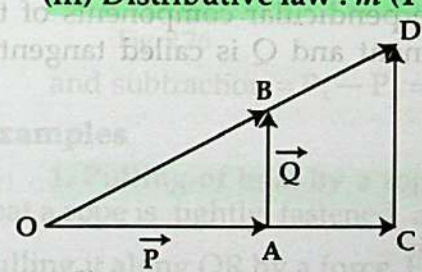


Fig. 2·22

Suppose $\vec{OA} = \vec{P}$ and $\vec{AB} = \vec{Q}$ [Fig. 2·22]. Let us connect OB. Now from the law of triangle,

$$\vec{OA} + \vec{AB} = \vec{OB} = \vec{P} + \vec{Q}$$

Let us take two points C and D on the extended lines of OA and OB so that

$$\vec{OC} = m\vec{OA} = m\vec{P} \text{ and } \vec{CD} = m\vec{AB} = m\vec{Q}$$

Now, since ΔOAB and ΔOCD are right angle triangles, we get,

$$\frac{\vec{OC}}{\vec{OA}} = \frac{\vec{OD}}{\vec{OB}} = \frac{\vec{CD}}{\vec{AB}} = \frac{m\vec{Q}}{\vec{Q}} = m$$

$$\therefore \vec{OD} = m\,\vec{OB} = m(\vec{P} + \vec{Q}) \quad \ldots \quad \ldots \quad \ldots \quad (2.11)$$

Again, according to the law of triangle,

$$\vec{OD} = \vec{OC} + \vec{CD} = m\vec{P} + m\vec{Q}$$

$$\therefore m(\vec{P} + \vec{Q}) = m\vec{P} + m\vec{Q} \quad \ldots \quad \ldots \quad \ldots \quad (2.12)$$

This is the distributive law. That means, vector addition follows the distribution law.

2·6 Vector addition and subtraction in terms of normal components

A vector quantity can be resolved by parallelogram law into two or more vectors in different directions. **The process of resolving a vector into two or more vectors is called resolution of a vector or vector resolution.** A vector quantity R can be resolved into perpendicular components and vector addition and subtraction can be done with it. **Each resolving vector is called component of the original vector.**

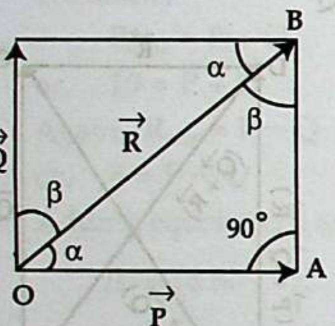


Fig. 2·23

Resolution into perpendicular components :

Let $\vec{R}$ be the vector along OB, so that $\vec{OB} = \vec{R}$. Now taking OB as diagonal let us draw the rectangle OABC [Fig. 2·23]. Here the components P and Q are perpendicular to each other, i,e., $\alpha + \beta = 90°$

$\therefore \sin(\alpha + \beta) = \sin 90° = 1$ and

$\sin\beta = \sin(90° - \alpha) = \cos\alpha$

According to law of traingle of trigonometry we get from the triangle OAB,

$$\frac{OA}{\sin\beta} = \frac{AB}{\sin\alpha} = \frac{OB}{\sin 90°}$$

$$\frac{P}{\cos\alpha} = \frac{Q}{\sin\alpha} = \frac{R}{\sin 90°}$$

$\therefore P = R\cos\alpha$ and $Q = R\sin\alpha$ ... ... (2.13)

The two components P and Q are called the perpendicular components of the rincipal vector R. P is called the horizontal component and Q is called tangential mponent.

Vector form of the two components are—

$\vec{P} = R\cos\alpha\,\hat{i} + R\sin\alpha\,\hat{i}$

So, **their vector addition,**

$\vec{R} = \vec{P} + \vec{Q} = R\cos\alpha\,\hat{i} + R\sin\alpha\,\hat{i}$ ... ... (2.14)

and vector subtraction will be the vector subtraction of $\vec{P}$ and $\vec{Q}$.

Addition and subtraction can be explained by the help of three arms of a triangle. Suppose $\vec{P}$ is acting along the arm AB of a triangle and $\vec{Q}$ is acting along the arm BC [Fig. 2·24, solid lines]. Then their vector addition, $\vec{R} = \vec{P} + \vec{Q}$ can be represented by the third arm AC.

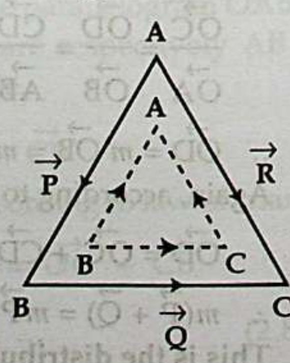


Fig. 2·24

Again, if $\vec{P}$ and $\vec{Q}$ are acting along BA and BC the subtraction of the vector quantities, $\vec{R} = \vec{P} - \vec{Q}$ can be represented by the third arm CA [Fig. 2·24, hypen lines].

Work : Look at the figure 2·25. The car is descending by standing off the engine. Only the weight of the car is active downward. Neglecting friction explain the two components one along the inclined plane and the other along the perpendicular direction.

Weight *mg* of the body acts vertically downward. *mg* can be resolved into two components—one along the inclined plane and the other one along the perpendicular direction of the plane. The magnitudes of the components, according to the figure, are $mg \cos\theta$ and $mg \sin\theta$ respectively. Here θ is the angle of inclination of the plane. $mg \cos\theta$ becomes balanced by the normal reaction of the inclined plane. The car descends only due to the force $mg \sin\theta$.

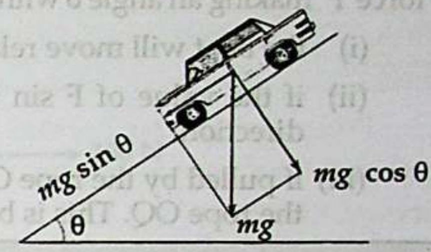


Fig. 2·25

Example : A 30 N force is inclined at 60° angle with Y-axis. Calculate the perpendicular components of the force along X and Y-axes and find their addition and subtraction values.

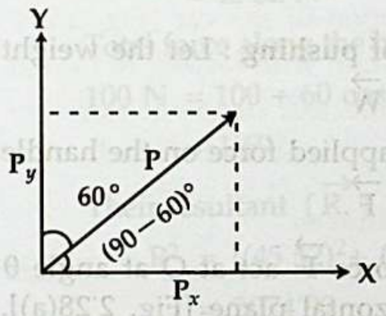


Fig. 2·26

Let force P = 30 N. Its components along X and Y-axes are P_x and P_y respectively. According to the principle of equal angle of vector, we get

$$P_x = P \sin 60° = 30 \times \frac{\sqrt{3}}{2} \text{ N} = 15\sqrt{3} \text{ N}$$

$$P_y = P \cos 60° = 30 \times \frac{1}{2} \text{ N} = 15 \text{ N}$$

$$\text{Their addition} = P_x + P_y = (15\sqrt{3} + 15) \text{ N} = 15(\sqrt{3}+1) \text{ N}$$

and subtraction $= P_x - P_y = (15\sqrt{3} - 15) \text{ N} = 15(\sqrt{3} - 1) \text{ N}$

Examples

1. Pulling of boat by a rope/Towing a boat : Let M be a boat. If at point O of the boat a rope is tightly fastened and the boat is moving by the side of the river due to pulling it along OR by a force $\vec{F}$. Now at point O the applied force $\vec{F}$ can be resolved into components, viz., horizontal and vertical components.

The horizontal component of the force = $F \cos\theta$ and its direction is along OA [Fig. 2·27].

The vertical component = $F \sin\theta$ and is acting along OB.

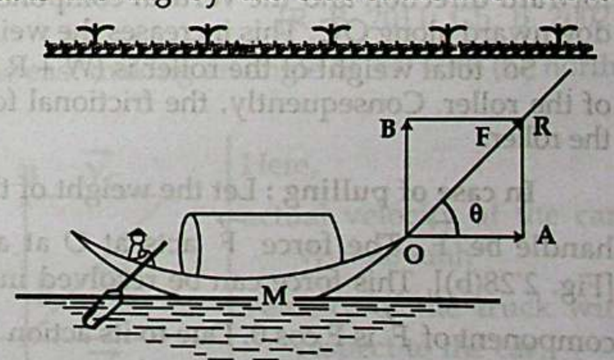


Fig. 2·27

The horizontal component of the force $F \cos\theta$ pulls the boat in the forward direction and the vertical component of the force $F \sin\theta$ pulls the boat towards the edge of the river. But by the rudder or helm of the boat vertical component $F \sin\theta$ is repulsed. The larger the length of the rope, the smaller is the value of θ; so smaller is the magnitude of F sin θ, larger is the value of F cos θ. As a result the boat will move fast in the forward direction. i.e., if the rope of a tow is long then the boat will move fast. Again if by fastening the rope at point P and the boat is pulled forward against the current with a force $\vec{F}$ making an angle θ with the horizontal and if the length of the rope is OP then—

(i) the boat will move relatively faster,

(ii) if the value of F sin θ is less then the boat will move fast in the forward direction,

(iii) if pulled by the rope OP then motion of the boat will be slower than pulling by the rope OQ. This is because that length of OQ is longer and θ′ > θ.

Perceptional work : Why the handle of a trolley bag is made longer.

Example 2. Moving a Lawn Roller : When an object is either pulled or pushed on a plane, a frictional force acts between the plane and the object. This retards the motion of the object. The heavier the object, larger is the frictional force. A lawn roller is made moving either by pushing or pulling .

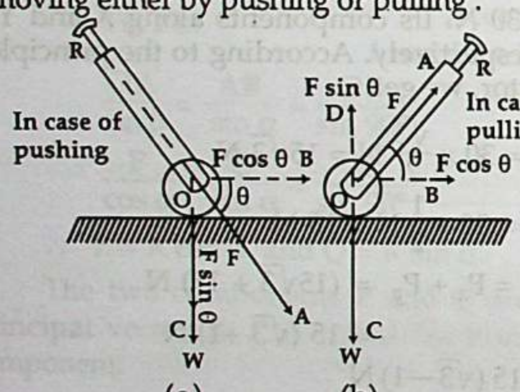


Fig. 2·28

In case of pushing : Let the weight of the roller be $\vec{W}$ and the applied force on the handle of the roller be $\vec{F}$.

Let the force $\vec{F}$ act at O at angle θ with the horizontal plane [Fig. 2·28(a)]. Now, $\vec{F}$ can be resolved at O into two normal components.

The horizontal component of the force = F cos θ, which acts along OB in the forward direction and the vertical component of the force = F sin θ, whose direction is downward along OC. This increases the weight of the roller.

So, total weight of the roller is (W + R sin θ), which is more than the actual weight of the roller. Consequently, the frictional force also increases. So, it is difficult to push the roller.

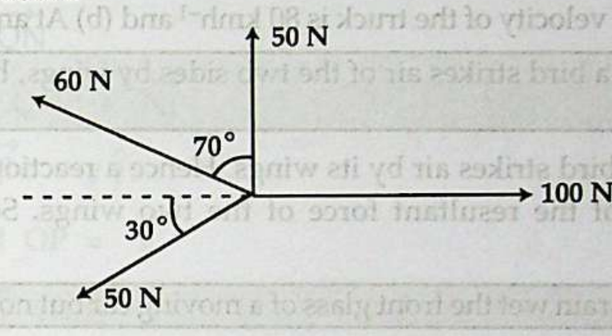
In case of pulling : Let the weight of the roller be $\vec{W}$ and the applied force on the handle be $\vec{F}$. The force $\vec{F}$ acts at O at an angle θ along the horizontal line OB [Fig. 2·28(b)]. This force can be resolved into two normal components. The horizontal component of $\vec{F}$ is F cos θ. Due to its action the roller moves in the forward direction.

The vertical component of $\vec{F}$ is F sin θ, which acts upward along OD. So the total weight of the roller decreases. The weight of the roller is (W — F sin θ), which is less than the actual weight of the roller. The frictional force also decreases, so it becomes easier to pull a roller.

So, it can be said that, **it is easier to pull a lawn roller than to push.**

Mathematical examples

1. In the figure below, determine the resultant of the forces along the directions of 50 N and 100N.



Total force along the perpendicular direction of the force

$50\text{ N} = 50 + 60 \sin(90° + 70°) + 50 \sin(180° + 30°)$

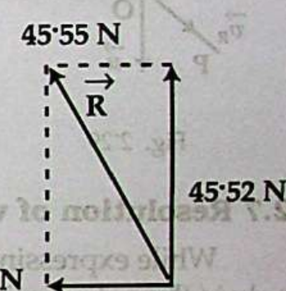
$= 45{\cdot}52\text{ N}$

Total force along the horizontal direction of the force

$100\text{ N} = 100 + 60 \cos(90° + 70°) + 50 \cos(180° + 30°)$

$= 1{\cdot}69$

Their resultant ($\vec{R}$) is shown in the figure,



$R^2 = \{(45{\cdot}52)^2 + (1{\cdot}69)^2\}$

$= 2074{\cdot}92$

$\therefore\ R = 45{\cdot}55\text{ N}$

2. The driver of a car moving with velocity of 40 km per hour in the east saw a truck moving in the north with velocity of $40\sqrt{3}$ km per hour. (a) what is the actual velocity of the truck ? and (b) in which direction the truck is moving ?

[R. B. 2011; Ch. B. 2002]

(a) Suppose, the truck is moving in the east making an angle of 30° with the north. According to the law of triangle we get,

$\vec{V}_T = \vec{V}_{TC} + \vec{V}_C$

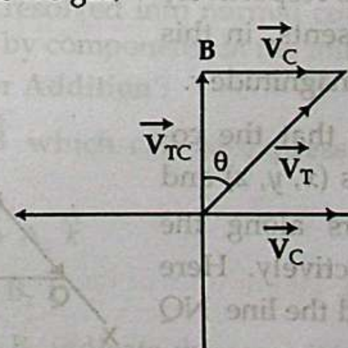
$\therefore V_T^2 = V_{TC}^2 + V_C^2$

or, $V_T = \sqrt{V_{TC}^2 + V_C^2}$

$= \sqrt{(40\sqrt{3})^2 + (40)^2}$

$= \sqrt{40^2(4)} = 2 \times 40$

$= 80\text{ kmh}^{-1}$



Here,

actual velocity of the car, $V_C = 40\text{ kmh}^{-1}$

velocity of the truck with respect to the car, $V_{TC} = 40\sqrt{3}\text{ kmh}^{-1}$

actual velocity of the truck, $V_T = ?$

(b) Again, $\tan\theta = \frac{V_C}{V_{TC}} = \frac{40}{40\sqrt{3}} = \frac{1}{\sqrt{3}} = \tan 30°$

$\therefore \quad \theta = 30°$

Ans. (a) Actual velocity of the truck is 80 kmh^{-1} and (b) At an angle of 30° in the east.

Work : While flying a bird strikes air of the two sides by wings, but how the bird moves forward ?

While flying a bird strikes air by its wings. Hence a reaction force is created in the opposite direction of the resultant force of the two wings. So, the bird flies in the forward direction.

Work : Why drops of rain wet the front glass of a moving car but not the back side glass?

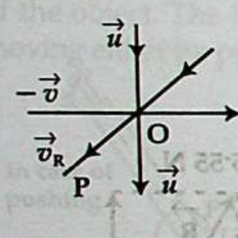


Fig. 2·29

Let the velocity of the car be $\vec{v}$ and that of rain is $\vec{u}$.

$\therefore$ resultant velocity, $\vec{v}_R = \vec{u} + (-\vec{v})$ acts along OP, i.e., acts along the direction of motion of the car [Fig. 2·29]. Here direction of relative velocity of rain with respect to the car acts obliquely towards the front side. So, drops of rain wet glass of the front side, instead of the glass of the back side.

2.7 Resolution of vector in three dimensional co-ordinates

While expressing a vector quantity with its components we will consider resolution only in three dimensional rectangular coordinates.

In three dimensional coordinate system a position vector can be expressed in the following way which is considered as the resolution of the vector in three rectangular coordinates.

$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$. Here position coordinate of P is (x, y, z).

Proof : Let OX, OY and OZ be three lines perpendicular to one another and corresponding to X, Y and Z axes respectively [Fig. 2·30]. The line OP represents, in this system of axes, the vector $\vec{r}$ of magnitude r.

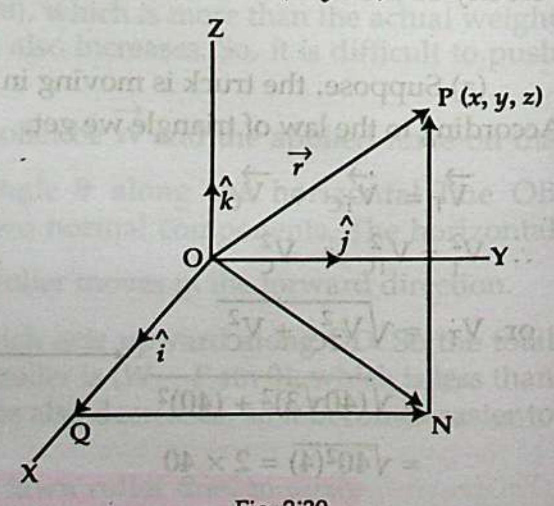


Fig. 2·30

Besides, let us consider that the co-ordinate of the end point of P is (x, y, z) and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors along the positive axes X, Y and Z respectively. Here PN is normal to the plane XY and the line NQ is normal to OX [Fig. 2·30].

Now from Fig. 2·30 we get by vector addition rules,

$$\vec{OP} = \vec{ON} + \vec{NP} \text{ and}$$

$$\vec{ON} = \vec{OQ} + \vec{QN}$$

$$\therefore \vec{OP} = \vec{OQ} + \vec{QN} + \vec{NP}$$

But, $\vec{OQ} = x\hat{i}$, $\vec{QN} = y\hat{j}$,

$$\vec{NP} = z\hat{k} \text{ and } \vec{OP} = \vec{r}$$

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \ldots \quad \ldots \quad \ldots \quad (2·15)$$

Here x, y and z are the components of vector $\vec{r}$ along X, Y and Z axes respectively and $\vec{r}$ is the position vector in three dimensional co-ordinate. Equation (2.15) is the required position vector.

Modulus of $\vec{r}$:

From fig. 2·30, we get

$$OP^2 = ON^2 + NP^2 \text{ and } ON^2 = OQ^2 + QN^2$$

$$\therefore \quad OP^2 = OQ^2 + QN^2 + NP^2$$

or, $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad r = \sqrt{x^2 + y^2 + z^2} \quad \ldots \quad \ldots \quad \ldots \quad (2·16)$$

Unit vector along or parallel to $\vec{r}$:

The unit vector along or parallel to $\vec{r}$ is,

$$\hat{r} = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{(x^2 + y^2 + z^2)}} \quad \ldots \quad \ldots \quad \ldots \quad (2·17)$$

Vector addition and subtraction of resolved normal components

If two or more vectors are resolved into normal components, then their addtion and subtraction can be expressed by components in the following way.

(a) Determination of Vector Addition :

Let two vectors $\vec{A}$ and $\vec{B}$ which can be expressed with the help of normal components as,

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

Here A_x, A_y, A_z and B_x, B_y, B_z indicate the magnitudes of the components of the two vectors $\vec{A}$ and $\vec{B}$ respectively along X, Y and Z axes.

Now, adding $\vec{A}$ and $\vec{B}$, we get,

$$\vec{A} + \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k} \quad \ldots \quad \ldots \quad (2.18)$$

Now if $\vec{A} + \vec{B} = \vec{R}$ and R_x, R_y and R_z are the magnitudes of R along X, Y and Z axes respectively, then

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

$$\therefore \vec{A} + \vec{B} = \vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

$$= R_x\hat{i} + R_y\hat{j} + R_z\hat{k} \quad \ldots \quad \ldots \quad \ldots \quad (2.19)$$

Modulus of the resultant :

From equations (2.18) and (2.19) we get,

$$R_x = A_x + B_x,\ R_y = A_y + B_y,\ R_z = A_z + B_z$$

$$\therefore |\vec{R}| = |\vec{A} + \vec{B}| = \sqrt{R_x^2 + R_y^2 + R_z^2}$$

$$= \sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}$$

Parallel unit vector $\hat{r}$ of $\vec{R}$ along $\vec{R}$ is

$$\hat{r} = \frac{\vec{R}}{|\vec{R}|} = \frac{R_x\hat{i} + R_y\hat{j} + R_z\hat{k}}{\sqrt{R_x^2 + R_y^2 + R_z^2}}$$

$$= \frac{(A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}}{\sqrt{(A_x + B_x)^2 + (A_y + B_y)^2 + (A_z + B_z)^2}}$$

(b) Determination of vector subtraction :

Subtraction of the vectors $\vec{A}$ and $\vec{B}$ can be determined in the following way.

$$\vec{A} - \vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) - (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$$

$$= (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k} \quad \ldots \quad \ldots \quad (2.20)$$

Now if the result of subtraction is $\vec{R}$, then

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

Here, R_x, R_y and R_z are magnitudes of the components of $\vec{R}$ along X, Y and Z axes.

$$\therefore \vec{A} - \vec{B} = \vec{R} = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}$$

$$= R_x\hat{i} + R_y\hat{j} + R_z\hat{k} \quad \ldots \quad \ldots \quad (2.21)$$

Modulus of the resultant :

From equations (2.20) and (2.21) we get,

$R_x = A_x - B_x,\ R_y = A_y - B_y,\ R_z = A_z - B_z$

$$\therefore\ |\vec{R}| = |\vec{A} - \vec{B}| = \sqrt{(A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2}$$
$$= \sqrt{R_x^2 + R_y^2 + R_z^2}$$

Parallel unit vector $\hat{r}$ of $\vec{R}$ along $\vec{R}$ is

$$\hat{r} = \frac{\vec{R}}{|\vec{R}|} = \frac{R_x\hat{i} + R_y\hat{j} + R_z\hat{k}}{\sqrt{R_x^2 + R_y^2 + R_z^2}} = \frac{(A_x - B_x)\hat{i} + (A_y - B_y)\hat{j} + (A_z - B_z)\hat{k}}{\sqrt{(A_x - B_x)^2 + (A_y - B_y)^2 + (A_z - B_z)^2}}$$

2·8 Multiplications of two vector quantities

Multiplications of two vector quantities are of two kinds, viz.—

(i) Scalar product or Dot product

(ii) Vector product or Cross product

These two products or multiplications are discussed below separately.

2·8·1 Scalar product or Dot product of vectors

The scalar product or dot product of two vectors is a scalar quantity equal to the product of the magnitudes of the two vectors and the cosine of the angle between them. It is expressed by inserting a dot (·) sign between the two vector quantities and read as 'first quantity dot second quantity'. So the other name of this product is dot product.

The scalar quantity that is obtained due to the product of the two vector quantities is called the scalar product of the two vectors.

Or, the product of the magnitudes of the two vectors when multiplied by their cosine of the angle between them, then that product is called the scalar product or dot product.

Explanation : Let $\vec{P}$ and $\vec{Q}$ represent two vectors. Arrow marked straight lines OA and OC indicate magnitude and direction of the two quantities [Fig. 2·31]. These are inclined to each other at an angle α. $\vec{P}.\vec{Q}$ indicates their scalar or dot product and read as P dot Q. According to the definition dot product of the two vectors is,

$\vec{P}.\vec{Q} = |\vec{P}||\vec{Q}| \cos\alpha,\ \ \pi \geq \alpha \geq 0$

or, $\vec{P}.\vec{Q} = PQ\cos\alpha = QP\cos\alpha$ ... (2·22)

Here $0 \leq \alpha \leq \pi$

But Q cos α is the component of $\vec{Q}$ in the direction of $\vec{P}$ or projection of $\vec{Q}$ on $\vec{P}$ and P cos α is the component of $\vec{P}$ in the direction of $\vec{Q}$ [Fig. 2·31].

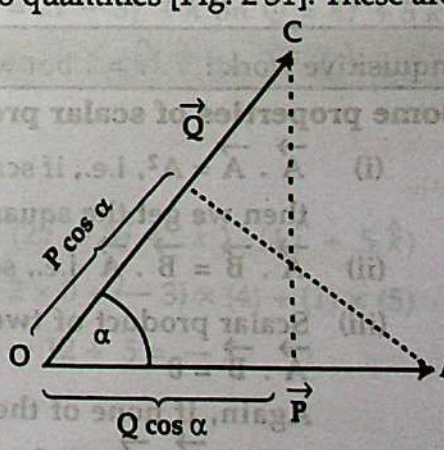


Fig. 2·31

Again, $\vec{P} . \vec{Q} = PQ \cos\alpha = Q(P\cos\alpha)$

But $P\cos\alpha$ is the component of $\vec{P}$ in the direction of $\vec{Q}$ or normal projection of $\vec{P}$ on $\vec{Q}$.

So, scalar product of any two vectors means the product of the modulus or magnitude of a vector and component of the other vector in the direction of the first one or projection of the other vector on the first vector.

It is seen from equation (2.22) that the product is a scalar quantity.

Special cases :

(a) If $\alpha = 0°$, then $\vec{P} . \vec{Q} = PQ \cos 0° = PQ$. In this case the two vectors will be parallel to each other.

(b) If $\alpha = 90°$, then $\vec{P} . \vec{Q} = PQ \cos 90° = 0$. In this case the two vectors will be perpendicular to each other.

(c) If $\alpha = 180°$, then $\vec{P} . \vec{Q} = PQ \cos 180° = - PQ$. In this case the two vectors will be parallel to each other but oppositely directed.

[N.B. $\vec{P} . \vec{Q} = PQ\cos\alpha = P \times Q \cos\alpha = Q \times P \cos\alpha$. Here, $Q\cos\alpha$ = the projection of Q along $\vec{P}$ and $P\cos\alpha$ = the projection of $\vec{P}$ along $\vec{Q}$].

Example of scalar product: Both force $\vec{F}$ and displacement $\vec{s}$ are vector quantities but their scalar product work (W) is a scalar quantity i.e.,

$$W = \vec{F} . \vec{s} = Fs\cos\alpha \quad \dots \quad \dots \quad \dots \quad (2.23)$$

Potential energy, electric potential etc are also examples of scalar product.

According to the rules of scalar multiplication

(i) $\vec{P} . \vec{Q} = \vec{Q} . \vec{P}$

(ii) $\hat{i} . \hat{i} = \hat{j} . \hat{j} = \hat{k} . \hat{k} = 1$

(iii) $\hat{i} . \hat{j} = \hat{j} . \hat{k} = \hat{k} . \hat{i} = 0$

Inquisitive work : $\hat{i} . \hat{i} = 1$ but why $\hat{i} . \hat{j} = 0$?

Some properties of scalar product

(i) $\vec{A} . \vec{A} = A^2$, **i.e., if scalar product is done by taking the same vector twice, then we get the square of the magnitude of that vector.**

(ii) $\vec{A} . \vec{B} = \vec{B} . \vec{A}$ **i.e., scalar product obeys commutative law.**

(iii) **Scalar product of two mutually perpendicular vectors becomes zero. i.e.,** $\vec{A} . \vec{B} = 0$

Again, if none of the magnitude of the two vectors is zero ($A \neq 0$, $B \neq 0$), then if $\vec{A} . \vec{B} = 0$ then $\vec{A} \perp \vec{B}$.

(iv) **Angle between the two vectors, $\theta = \cos^{-1}\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right)$**

(v) **Scalar products of unit vectors of equal angles $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$**

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{j} = \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0$

(vi) **Scalar product of two vectors by components $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$**

> Work : A displacement $\vec{r} = (3\hat{i} + 5\hat{j} - \hat{k})$ m is produced due to the application of force $\vec{F} = (5\hat{i} + 3\hat{j} + 2\hat{k})$ N on a body.
>
> Hints : Work, $W = \vec{F}\cdot\vec{r}$ is to be determined.

Mathematical examples

1. Find the scalar product of the two vectors $\vec{A} = 9\hat{i} + \hat{j} - 6\hat{k}$ and $\vec{B} = 4\hat{i} - 6\hat{j} + 5\hat{k}$ and also show that they are perpendicular to each other.

[J. B. 2011; B. B. 2009; D. B. 2004; R. B. 2001]

We know, if $\vec{A}\cdot\vec{B} = 0$ then the vector quantities will be perpendicular to each other.

According to question, it is required that $\vec{A}\cdot\vec{B} = AB\cos\theta = 0$

$\vec{A}\cdot\vec{B} = A_x B_x + A_y B_y + A_z B_z$

$\therefore\ \vec{A}\cdot\vec{B} = (9\hat{i} + \hat{j} - 6\hat{k})\cdot(4\hat{i} - 6\hat{j} + 5\hat{k})$

$= 9\times 4 + (1\times -6) + (-6\times 5)$

$= 36 - 6 - 30 = 0$

As $\vec{A}\cdot\vec{B} = 0$, but $A \neq 0$ and $B \neq 0$

$\therefore\ \cos\theta = 0 = \cos 90°$

So the two vectors are perpendicular to each other.

2. Determine the normal projection of vector $\vec{P} = 2\hat{i} - 3\hat{j} + \hat{k}$ on $\vec{Q} = 4\hat{j} + 5\hat{k}$.

If θ is the angle between $\vec{P}$ and $\vec{Q}$, then normal projection of $\vec{P}$ on $\vec{Q} = Q\cos\theta$

We know, $\vec{P}\cdot\vec{Q} = PQ\cos\theta$

$\therefore\ Q\cos\theta = \frac{\vec{P}\cdot\vec{Q}}{P} = \frac{\vec{P}\cdot\vec{Q}}{|\vec{P}|}$

and $|\vec{P}| = \sqrt{P_x^2 + P_y^2 + P_z^2}$

$= \sqrt{(2)^2 + (-3)^2 + (1)^2}$

$= \sqrt{4+9+1} = \sqrt{14}$

$\therefore\ Q\cos\theta = \frac{-7}{\sqrt{14}}$

Here,

$\vec{P}\cdot\vec{Q} = (2\hat{i} - 3\hat{j} + \hat{k})\cdot(4\hat{j} + 5\hat{k})$

$= 2\times 0 + (-3)\times(4) + (1)\times(5)$

$= -12 + 5 = -7$

3. For what value of m the vectors $\vec{A} = 2\hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{B} = m\hat{i} + 3\hat{j} + 4\hat{k}$ will be perpendicular to each other ?

We know, if two vectors are mutually perpendicular, then

$\vec{A}.\vec{B} = AB \cos\theta = AB \cos 90° = 0$

$\therefore \quad \vec{A}.\vec{B} = (2\hat{i} + 2\hat{j} - 2\hat{k}).(m\hat{i} + 3\hat{j} + 4\hat{k})$

or, $2m + 6 - 8 = 0$

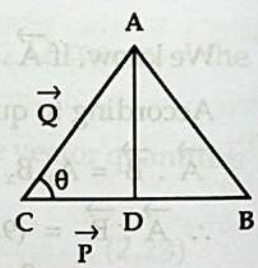
or, $2m - 2 = 0$

$\therefore \quad m = 1$

4. If two arms of a triangle are represented by $\vec{P}$ and $\vec{Q}$, then show that the area of the triangle $= \frac{1}{2} |\vec{P} \times \vec{Q}|$

In the triangle ABC, $\vec{CB} = \vec{P}$

$\vec{CA} = \vec{Q}$



and angle between them is θ (see the Fig.)

height of the triangle, AD $= AC \cos\theta = Q \sin\theta$

$\therefore$ area of the triangle $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}$ (BC) (AD)

$= \frac{1}{2} PQ \sin\theta = \frac{1}{2} |\vec{P} \times \vec{Q}|$

2·8·2 Vector product or Cross product of vectors

If the product of two vectors is a vector quantity, then that product is called vector product or cross product. The magnitude of the cross product is obtained by multiplying the magnitudes of their constituent vectors with the sine of the angle between them. In order to execute vector products of two vectors a cross (×) sign is added between them. For this reason, the other name of this product is cross product. The direction of the cross product is determined by the right hand screw rule.

Explanation : Let $\vec{P}$ and $\vec{Q}$ be two vectors mutually acting at O making an angle α. According to definition the vector product is,

$\vec{R} = \vec{P} \times \vec{Q} = \hat{\eta}\ |\vec{P}|\ |\vec{Q}| \sin\alpha,\ 0 \leq \alpha \leq \pi$ ... ... [2·24(a)]

here η represents the direction of the product [Fig. 2·32]

or, $\vec{R} = \vec{Q} \times \vec{P} = \hat{\eta}\ QP \sin\alpha,\ 0 \leq \alpha \leq \pi$ ... ... [2·24(b)]

Here $\hat{\eta}$ (eta) is a unit vector which represents the direction of $\vec{R}$ [Fig. 2·33]. The direction $\hat{\eta}$ is found from the **right-handed screw rule.**

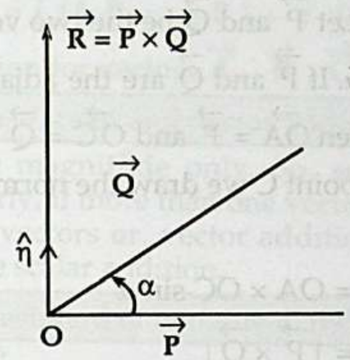


Fig. 2·32

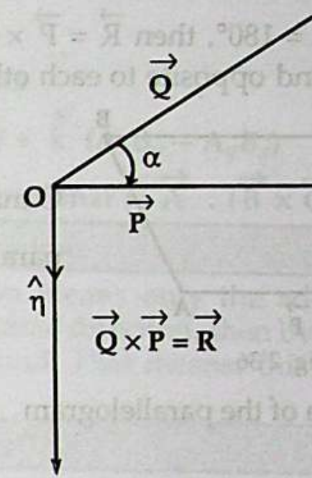


Fig. 2·33

Right handed screw rule : By keeping a right handed screw on the plane that contains the two vectors and if the first vector is rotated towards the second vector along the smallest angle, then the direction through which the screw rotates will be the direction of $\vec{R}$, that means, the direction of $\hat{\eta}$.

According to the above rule the direction of $\vec{P} \times \vec{Q}$ will be upward [Fig. 2·34] and that of $\vec{Q} \times \vec{P}$ will be downward [Fig. 2·35]. That means, in the first case the direction of the right handed screw will be anticlockwise and in the second case it is clockwise. Anticlockwise direction is considered as positive, whereas clockwise direction is considered as negative.

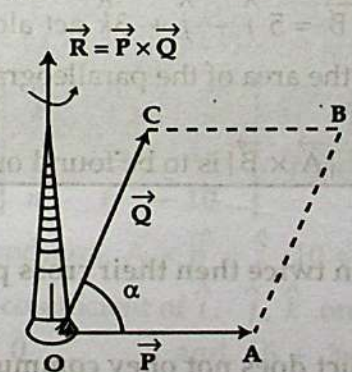


Fig. 2·34

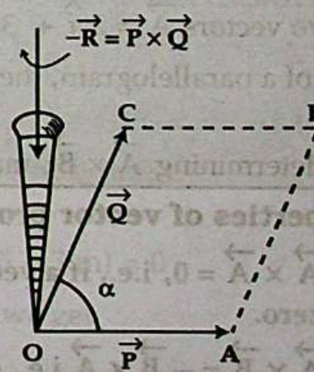


Fig. 2·35

Special Cases : (a) If $\alpha = 0°$, then $\vec{R} = \vec{P} \times \vec{Q} = \hat{\eta}\, PQ \sin 0° = 0$; in this case the two vectors will be mutually parallel to each other.

(b) If $\alpha = 90°$, then $\vec{R} = \vec{P} \times \vec{Q} = \hat{\eta}$ PQ sin 90° = PQ; in this case two vectors will be mutually perpendicular to each other.

(c) If $\alpha = 180°$, then $\vec{R} = \vec{P} \times \vec{Q} = \hat{\eta}$ PQ sin 180° = 0; in this case, two vectors will be parallel and opposite to each other.

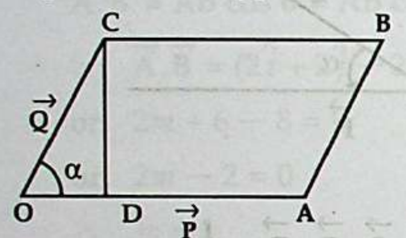


Fig. 2·36

Example : Let $\vec{P}$ and $\vec{Q}$ be the two vectors acting at O making angle α. If $\vec{P}$ and $\vec{Q}$ are the adjacent sides of a parallelogram, then $\vec{OA} = \vec{P}$ and $\vec{OC} = \vec{Q}$ [Fig. 2·36].

Now from point C we draw the normal CD on OA.

∴ Area of the parallelogram $= OA \times CD = OA \times OC \sin\alpha$

$= PQ \sin\alpha = |\vec{P} \times \vec{Q}|$

Conclusion : From the above result it is concluded that the area of a parallelogram formed by two vectors as adjacent sides is equal to the magnitude of vector product of the two vectors.

According to the rules of vector product,

(i) $\vec{P} \times \vec{Q} = -\vec{Q} \times \vec{P}$

(ii) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(iii) $\hat{i} \times \hat{j} = \hat{k} = -(\hat{j} \times \hat{i})$

(iv) $\hat{j} \times \hat{k} = \hat{i} = -(\hat{k} \times \hat{j})$

(v) $\hat{k} \times \hat{i} = \hat{j} = -(\hat{i} \times \hat{k})$

Work : If two vectors $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{B} = 5\hat{i} + \hat{j} + 3\hat{k}$ act along two adjacent sides of a parallelogram, then calculate the area of the paralleogram.

[Ans. 7·64 units]

Hints : By determining $\vec{A} \times \vec{B}$, magnitude of $|\vec{A} \times \vec{B}|$ is to be found out.

Some properties of vector product

(i) $\vec{A} \times \vec{A} = 0$, i.e., if a vector is taken twice then their cross product becomes zero.

(ii) $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ i.e., cross-product does not obey communicative law.

(iii) If $\vec{A} \perp \vec{B}$, then magnitude of $\vec{A} \times \vec{B} = |\vec{A} \times \vec{B}|$ = AB sin 90° = AB.

These three vectors $\vec{A}$, $\vec{B}$ and $\vec{A} \times \vec{B}$ are mutually perpendicular to one another.

(iv) **Vector products of unit vectors of equal angles,**

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i},\ \hat{i} \times \hat{k} = -\hat{j}$

(v) **Vector product of two vectors by components**

$$\vec{A} \times \vec{B} = \hat{i}\,(A_xB_z - A_zB_x) + \hat{j}\,(A_zB_x - A_xB_z) + \hat{k}\,(A_xB_y - A_yB_x)$$

(vi) **Condition for vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ to become coplanar is $\vec{A} \cdot (\vec{B} \times \vec{C}) = 0$**

Work : In which case scalar and vector additions become same.

Scalars have magnitude only. So, scalar addition means only the addition of magnitude. Similarly, if more than one vector are in the same direction, then by addition the magnitude of vectors or, vector addition can be found. That means, this addition will be equal to the scalar addition.

Inquiry : Can the resultant of two unequal vectors be zero ?

Let $\vec{P}$ and $\vec{Q}$ be two vectors and angle between them $= \theta$

Magnitude of their resultant, $R = \sqrt{P^2 + Q^2 + 2PQ\cos\theta}$

When $\theta = 180°$, then R will be minimum. That means, $R_{min} = P - Q$

Now, it is seen that if P and Q are equal then R will be zero. But if P and Q are unequal, then the minimum value of R cannot be zero. So, the resultant of two unequal vectors can never be zero.

Do yourself : If the angle between $\vec{A}$ and $\vec{B}$ is 45°, then show that $\vec{A} \cdot \vec{B} = |\vec{A} \times \vec{B}|$

Mathematical examples

1. $\vec{A} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{B} = m\hat{i} + 6\hat{j} - 10\hat{k}$. For what value of m the two vectors will be parallel to each other ? [B. B. 2007; J. B. 2007]

We know, if $\vec{A} \times \vec{B} = 0$, then vectors $\vec{A}$ and $\vec{B}$ will be parallel to each other.

Here $\vec{A} \times \vec{B} = (\hat{i} - 3\hat{j} + 5\hat{k}) \times (m\hat{i} + 6\hat{j} + 10\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 5 \\ m & 6 & -10 \end{vmatrix} = \hat{i}\,(30 - 30) - \hat{j}\,(-10 - 5m) + \hat{k}\,(6 + 3m) = \hat{j}\,(10 + 5m) + \hat{k}\,(6 + 3m)$$

According to condition, $\vec{A} \times \vec{B} = \hat{j}\,(10 + 5\text{ m}) + \hat{k}\,(6 + 3\text{ m}) = 0$

Comparing the co-efficient of $\hat{i}$, $\hat{j}$, $\hat{k}$ on both sides we get,

$10 + 5m = 0$ and $6 + 3m = 0$

or, $5m = -10$ or, $3m = -6$

$\therefore m = -\dfrac{10}{5} = -2$ $\quad \therefore m = -\dfrac{6}{3} = -2$

So, for $m = -2$, the vectors will be parallel to each other.

2. If $\vec{P} = 2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\vec{Q} = -2\hat{j} + \hat{i} + 3\hat{k}$, find a unit vector perpendicular to the plane of $\vec{P}$ and $\vec{Q}$. [J. B. 2009, 2006, 2004; Ch. B. 2008; C. B. 2008]

$\vec{P} \times \vec{Q}$ is a vector which is normal to the plane of $\vec{P}$ and $\vec{Q}$.

$$\therefore \vec{P} \times \vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -4 \\ 1 & -2 & 3 \end{vmatrix} = (9-8)\,\hat{i} + (-4-6)\,\hat{j} + (-4-3)\hat{k}$$

$$= \hat{i} - 10\hat{j} - 7\hat{k}$$

and $|\vec{P} \times \vec{Q}| = \sqrt{(1)^2 + (-10)^2 + (-7)^2}$

$$\therefore \; n = \frac{\vec{P} \times \vec{Q}}{|\vec{P} \times \vec{Q}|} = \frac{\hat{i} - 10\hat{j} - 7\hat{k}}{\sqrt{(1)^2 + (-10)^2 + (-7)^2}}$$

$$= \frac{\hat{i} - 10\hat{j} - 7\hat{k}}{\sqrt{1 + 100 + 49}} = \frac{\hat{i} - 10\hat{j} - 7\hat{k}}{\sqrt{150}}$$

3. If $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, find the angle between the vectors $\vec{A}$ and $\vec{B}$ and show that the two vectors are perpendicular to each other.

Let θ be the angle between $\vec{A}$ and $\vec{B}$

$\therefore \; |\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$

or, $|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2$

or, $(\vec{A} + \vec{B}) . (\vec{A} + \vec{B}) = (\vec{A} - \vec{B}) . (\vec{A} - \vec{B})$

or, $\vec{A} . \vec{A} + \vec{B} . \vec{A} + \vec{A} . \vec{B} + \vec{B} . \vec{B} = \vec{A} . \vec{A} - \vec{B} . \vec{A} - \vec{A} . \vec{B} + \vec{B} . \vec{B}$

or, $2(\vec{A} . \vec{B}) = -2(\vec{A} . \vec{B})$ $(\because \vec{A} . \vec{B} = \vec{B} . \vec{A})$

or, $4(\vec{A} . \vec{B}) = 0$

or, $\vec{A} . \vec{B} = 0$,

or, $AB \cos\theta = 0$, but $(A \neq 0 , B \neq 0)$

$\therefore \; \cos\theta = 0$ or, $\theta = 90°$

∴ angle between the two vectors is 90°; hence the two vectors are perpendicular to each other.

4. If the radius vector of a revolving particle is $\vec{r} = (2\hat{i} + 2\hat{j} - \hat{k})$m and applied force is $\vec{F} = (6\hat{i} + 3\hat{j} - 3\hat{k})$N, then find the magnitude and direction of the torque.

Here,
radius vector,
$\vec{r} = (2\hat{i} + 2\hat{j} - \hat{k})$ m
force, $\vec{F} = (6\hat{i} + 3\hat{j} - 3\hat{k})$ N
torque $\vec{\tau}$ = ?
magnitude of the torque, τ = ?

We know, $\vec{\tau} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 6 & 3 & -3 \end{vmatrix}$$

$$= \hat{i}(-6+3) - \hat{j}(-6+6) + \hat{k}(6-12)$$
$$= -3\hat{i} - 6\hat{k}$$

$\therefore \quad \vec{\tau} = -(3\hat{i} + 6\hat{k})$ N-m

magnitude of $\tau = \sqrt{(-3)^2 + (-6)^2}$
$= \sqrt{9+36} = \sqrt{45}$

Ans. $-(3\hat{i} + 6\hat{k})$ N-m, $\sqrt{45}$

5. Find the area of a parallelogram whose diagonals are $\vec{A} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{B} = \hat{i} - 3\hat{j} + 4\hat{k}$.

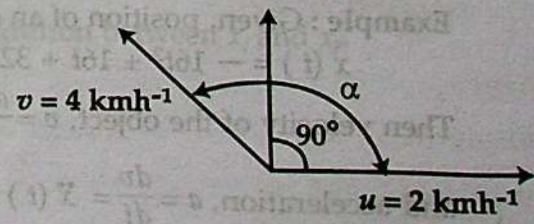
We know, area of a parallelogram $= \frac{1}{2} \times |\vec{A} \times \vec{B}|$

$$\therefore \quad \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \hat{i}(4-6) - \hat{j}(12+2) + \hat{k}(-9-1)$$
$$= -2\hat{i} - 14\hat{j} - 10\hat{k}$$

$$\therefore \quad \frac{1}{2} \times |\vec{A} \times \vec{B}| = \frac{1}{2}(\sqrt{(-2)^2 + (-14)^2 + (-10)^2}$$
$$= \frac{1}{2} \times \sqrt{300} = 8{\cdot}66$$

6. If there is no current in a river a swimmer can swim at velocity of 4 kmh^{-1}. If current of the river flows in a straight line at velocity of 2 kmh^{-1}, then in order to reach from one bank to exactly the opposite bank of the river in which direction the swimmer is to swim ?

Suppose, the velocity of current is u and that of the swimmer is v and angle between the two velocities is α. In order to cross the river strenight the swimer is to swim with resultant velocity R making angle $\theta = 90°$ with the velocity of the current.



We know,

$$\tan\theta = \frac{v\sin\alpha}{u + v\cos\alpha}$$

Here,
$u = 2\ \text{kmh}^{-1}$
$v = 4\ \text{kmh}^{-1}$
$\theta = 90°$
$\alpha = ?$

or, $\tan 90° = \dfrac{4\times\sin\alpha}{2 + 4\cos\alpha}$

or, $\infty = \dfrac{4\sin\alpha}{2 + 4\cos\alpha}$

or, $\dfrac{1}{0} = \dfrac{4\sin\alpha}{2 + 4\cos\alpha}$

or, $2 + 4\cos\alpha = 0$

or, $4\cos\alpha = -2$

or, $\cos\alpha = -\dfrac{1}{2}$

$\therefore\ \alpha = 120°$

That means, the swimmer is to swim making an angle of 120° with the current.

Calculate : If $\vec{A} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\vec{B} = \hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{C} = 2\hat{i} + 3\hat{j} - \hat{k}$ then prove whether the three vectors are coplanar or not.

We know,

If $\vec{A}\cdot(\vec{B}\times\vec{C}) = 0$, then the three vectors will be coplanar.

2·9 Calculus in physics

Calculus is the mathematical study of change. It has two important branches—**differential calculus and integral calculus.** Scientist Newton was the first to apply differential calculus in physics.

Importance : Calculus has great importance in physics. Many real processes are explained by equation containing derivatives. Those equations are called **differential equations.** Physics is associated with change of quantities and development with time. For practical knowledge of some important events idea of time derivative is essential. Particularly in Newtonian physics time derivative of position of a body is important.

Velocity is the time derivative of displacement.

Acceleration is the time derivative of velocity.

Uses : For determining velocity, acceleration, slope of a curved line etc differential calculus is applied. For calculating area, volume, centre of mass, work, pressure etc integral calculus is applied. Also acquiring knowledge about space, time and type of motion calculus is used.

Example : Given, position of an object on a straight line,

$$x(t) = -16t^2 + 16t + 32$$

Then velocity of the object, $v = \dfrac{dx(t)}{dt} = \dot{x}(t) = -32t + 16$

and acceleration, $a = \dfrac{dv}{dt} = \ddot{x}(t) = -32$

Newton's second law of motion can be expressed by differential equation in the following way,

$$F(t) = m\frac{d^2x}{dt^2}$$

Determination of work by variable force using integral calculus : Suppose, a variable force is acting along x-axis on a body. Magnitude of the force depends on distance x travelled by the body i.e., F is the function of distance x. In the figure a graph has been plotted showing the variation of $F(x)$ for different values of x.

Total displacement consists of equally divided n number of exceedingly small displacement Δx, where Δx is the displacement between x_i and $x_i + \Delta x$. For this small change of displacement the magnitude of force is considered constant and the value of force is F_1. So work done by this force for this small segment is, $\Delta W_1 = F_1 \Delta x$

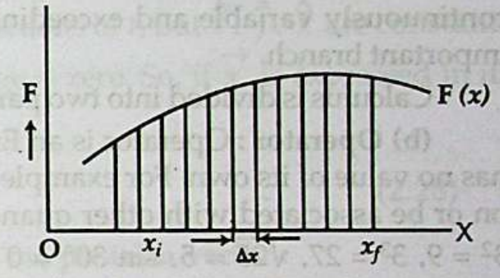


Fig. 2·37

Similarly, second segment is extended between $x_i + \Delta x$ to $x_i + 2\Delta x$ and the small displacement is Δx. For this small segment constant force is F_2. So work done by the force in the second segment is $\Delta W_2 = F_2 \, \Delta x$. So, total work done by the force $F(x)$ to move the body from position x_i to x_f is given by,

$$\begin{aligned} W &= \Delta W_1 + \Delta W_2 + \Delta W_3 + \ldots + \Delta W_N \\ &= F_1\Delta x + F_2\Delta x + F_3\Delta x + \ldots + F_N\Delta x \\ &= \sum_{K=1}^{N} F_K \, \Delta x \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.25) \end{aligned}$$

When Δx becomes smaller and smaller, that means the value of N becomes very large, calculated value of work done will be more correct. We can get correct value of work done by the force $F(x)$ if Δx approaches zero and N becomes infinity. Then correct value will be,

$$W = \lim_{\Delta x \to 0} \sum_{K=1}^{N} F_K \, \Delta x \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.26)$$

But in the language of calculus the quantity $\lim_{\Delta x \to 0} \sum_{K=1}^{N} F_K \, \Delta x$ is written as

$\int_{x_i}^{x_f} F(x)\, dx$, which indicates the integration between x_i and x_f.

So, equation (2·26) becomes,

$$W = \int_{x_i}^{x_f} F(x)\, dx \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.27)$$

Numerically this quantity is the area subtended by the curved line of force and the displacement between x_i and x_f. So, by integration work and area can be determined.

2·10 Vector calculus

Vector differentiation or vector derivatives

Before discussing vector differentiation or vector derivatives we need to know some important matter.

(a) Calculus : In scientific language calculus is a discipline of calculating continuously variable and exceedingly small fraction. In modern mathematics it is an important branch.

Calculus is divided into two parts— (1) **Differential calculus,** (2) **Integral calculus.**

(b) Operator : Operator is an English term. It is a mathematical symbol or index. It has no value of its own. For example, square, cube, root, sine, log etc. But when they act on or be associated with other quantities then they carry fixed values. As for example, $3^2 = 9$, $3^3 = 27$, $\sqrt{25} = 5$, $\sin 30° = 0.5$ etc. In other word $(10 \times)$ symbol does not have any value. But (10×5) has fixed value = 50. It means multiplication of 10 by 5. Now if $(10 \times)$ is designated by index C, then $10 \times 5 = C5$. So C is an operator. Integration is also an operator. Its symbol is $\int$ or Σ.

Definition : The mathematical symbol or index by which one quantity can be transformed into another quantity or can explain a variable quantity, then it is called an operator.

Differentiation is an operator with respect to t. This operator is $\frac{d}{dt}$, with respect to x it is $\frac{d}{dx}$, with respect to y it is $\frac{d}{dy}$ etc. The vector differention operator is designated by symbol $\vec{\nabla}$ (del) or ∇. Its other name is nabla. It is one type of mathematical representation. It can be written in different components as,

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Since both scalar and vector quantities can be differentiated, so differentiation is effective for both scalar and vector quantities.

Differentiation of a vector

Vector differential operator can be written by different components as—

$\frac{d}{dx}$, $\frac{d}{dy}$ and $\frac{d}{dz}$. Here $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ have no meaning. But when they act on x, y, z then they can be written as $\frac{dx}{dx}$, $\frac{dy}{dy}$ and $\frac{dz}{dz}$ which become meaningful. Now, if y is a quantity whose value depends on x i.e., operator $y(x)$ of yx, then y can be differentiated with respect to x and we get $\frac{dy}{dx}$. Here $\frac{dy}{dx}$ is the rate of change of y with respect to x for exceedingly small change of x. It is also called differential of y with respect to x. Again, when the value of Δt approaches zero, then for exceedingly small change of x, rate of change of x with respect to t is called the differential $\frac{dx}{dt}$. That means,

$$\lim_{\Delta u \to 0} \frac{\Delta \vec{x}}{\Delta t} = \frac{dx}{dt}$$

Now, if $\vec{x}$ is written in components as,

$\vec{x} = \hat{i}x_1 + \hat{j}y_1 + \hat{k}z_1$, here x_1, y_1, z_1 are the magnitudes of vector $\vec{x}$ respectively along X, Y and Z axes. Components x_1, y_1, z_1 are functions of t, but $\hat{i}$, $\hat{j}$, $\hat{k}$ are constants and do not change with time, i.e., their rate of change is zero. So, if $\vec{x}$ is expressed in its components then its differentiation is,

$$\frac{d\vec{x}}{dt} = \hat{i}\frac{dx_1}{dt} + \hat{j}\frac{dy_1}{dt} + \hat{k}\frac{dz_1}{dt} \quad \ldots \quad \ldots \quad \ldots \quad (2.28)$$

Derivation of velocity and acceleration from the position vector :

Let $\vec{r}$ be a position vector. Then

$$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$

The rate of change of $\vec{r}$ for exceedingly small time is called velocity, $\vec{v}$. So,

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} \quad \ldots \quad \ldots \quad \ldots \quad (2.29)$$

Again, the rate of change of velocity $\vec{v}$ for exceedingly small time is called acceleration $\vec{a}$.

$$\text{So, } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right)\hat{i} + \frac{d}{dt}\left(\frac{dy}{dt}\right)\hat{j} + \frac{d}{dt}\left(\frac{dz}{dt}\right)\hat{k}$$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k} \quad \ldots \quad \ldots \quad \ldots \quad (2.30)$$

General rule for differentiation of a scalar quantity is as follows :

(a) Firstly, the co-efficient of the variable quantity is to be multiplied by the power.

(b) Then '1' is to be subtracted from the power of the variable quantity.

Example : Let the distance, $s = 16t^2$. Here 16 is the co-efficient, t is the variable quantity and 2 is the power. According to the above rule, firstly 16 is to be multiplied by power 2 which gives 32 and subtraction of 1 from the power 2 of the variable quantity gives 1.

$$\therefore \quad \frac{ds}{dt} = v = 32t$$

Mathematical examples

1. How velocity and acceleration can be obtained from the differentiation of a position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$?

Here, position vector, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

We know, rate of change of r for exceedingly small time is called velocity. So,

$$\text{velocity, } \vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

For exceedingly small time rate of change of velocity $\vec{v}$ is called acceleration. So

$$\text{acceleration, } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$= \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j} + \frac{d^2z}{dt^2}\hat{k}$$

2. If two vectors $\vec{A} = \hat{i}t^2 - \hat{j}t + (2t+1)\hat{k}$ and $\vec{B} = 5\hat{i}t + \hat{j}t - \hat{k}t^3$, determine $\frac{d}{dt}(\vec{A} \cdot \vec{B})$ and $\frac{d}{dt}(\vec{A} \times \vec{B})$.

According to the question,

$$(\vec{A} \cdot \vec{B}) = \{\hat{i}t^2 - \hat{j}t + (2t+1)\hat{k}\} \cdot (5\hat{i}t + \hat{j}t - \hat{k}t^3)$$

$$= 5t^3 - t^2 - (2t+1)t^3$$

$$= 5t^3 - t^2 - 2t^4 - t^3$$

$$= 4t^3 - 2t^4 - t^2$$

$$\therefore \frac{d}{dt}(\vec{A} \cdot \vec{B}) = \frac{d}{dt}(4t^3 - 2t^4 - t^2)$$

$$= 12t^2 - 8t^3 - 2t$$

$$\text{Again, } (\vec{A} \times \vec{B}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & (2t+1) \\ 5t & t & -t^3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -t & (2t+1) \\ t & -t^3 \end{vmatrix} - \hat{j}\begin{vmatrix} t^2 & (2t+1) \\ 5t & -t^3 \end{vmatrix} + \hat{k}\begin{vmatrix} t^2 & -t \\ 5t & t \end{vmatrix}$$

$$= \hat{i}(t^4 - 2t^2 - t) - \hat{j}(-t^5 - 10t^2 - 5t) + \hat{k}(t^3 + 5t^2)$$

$$\therefore \frac{d}{dt}(\vec{A} \times \vec{B}) = \hat{i}(4t^3 - 4t - 1) + \hat{j}(5t^4 + 20t + 5) + \hat{k}(3t^2 + 10t)$$

3. If $\vec{A}$ is vector of constant magnitude then show that $\frac{d\vec{A}}{dt}$ is perpendicular to $\vec{A}$.

$$|\vec{A}| = \text{constant}$$

$$\therefore \quad \vec{A} \cdot \vec{A} = A^2 = \text{constant}$$

$$\therefore \quad \frac{d}{dt}(\vec{A} \cdot \vec{A}) = 0$$

$$\text{or,} \quad \vec{A} \cdot \frac{d\vec{A}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{A} = 0$$

$$\text{or,} \quad 2\vec{A} \cdot \frac{d\vec{A}}{dt} = 0 \;\; \text{or,} \;\; \frac{d\vec{A}}{dt} \cdot (\vec{A} + \vec{A}) = 0$$

So, the vector $\frac{d\vec{A}}{dt}$ remains perpendicular to the vector $\vec{A}$.

Integration

Suppose, a body is moving in a particular direction, i.e., it is moving along a straight line. Although the direction of velocity is fixed, but its magnitude is varying. Let, the magnitude of velocity v is dependent on time t. That means, v is the function of t. It is indicated by $v(t)$.

Let initial time be t_A and final time be t_B. We will find the distance travelled by the body during this time interval. Let us divide the time interval $t_B - t_A$ into exceedingly small N number of equal section Δt. Suppose the time interval from the initial time t_A to first small section $t_A + \Delta t$ is Δt. This time interval is so small that we can consider that the value of $v(t)$ is almost constant. Let the constant value of $v(t)$ be v. If the distance travelled in time Δt is Δs_1 then we get,

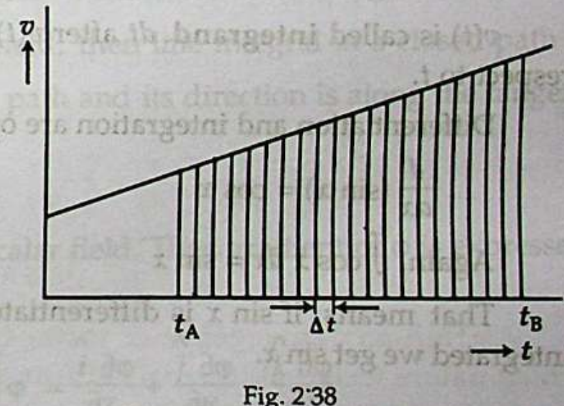


Fig. 2·38

$$\Delta s_1 = v_1 \Delta t$$

Similarly for the second, third sections time interval is Δt and the distance travelled will be respectively $\Delta s_2 = v_2\Delta t$ and $\Delta s_3 = v_3\Delta t$. For the Nth section, distance travelled is $\Delta s_N = v_N\Delta t$. Now, distance travelled between the time interval t_A and t_B will be the summation of $\Delta s_1, \Delta s_2, \Delta s_3 \ldots\ldots\ldots \Delta s_N$.

$$\text{So,} \quad s = \Delta s_1 + \Delta s_2 + \Delta s_3 + \ldots + \Delta s_N$$

$$\text{or,} \quad s = v_1\Delta t + v_2\Delta t + v_3\Delta t + \ldots + v_N\Delta t$$

$$\text{or,} \quad s = \sum_{n=1}^{N} v_n \Delta t \qquad \ldots \qquad \ldots \qquad \ldots \qquad (2.31)$$

It is to be noted that here we have assumed that the value of velocity is constant for small time interval Δt. If the value of v would have been exactly constant, then we could get more accurate value of distance from the equation 2·33. Now as the time interval becomes small to smaller, value of v will be close to closer to the constant value. We can get the accurate value of distance if Δt tends to zero and number of sections N becomes infinity. Then we can write,

$$s = \lim_{\Delta t \to 0} \sum_{n=1}^{N} v_n \Delta t$$

In calculus, the quantity $\lim\limits_{\Delta t \to 0} \sum\limits_{n=1}^{N}$ is written as $\int_{t_A}^{t_B} v(t)dt$.

The quantity $\int_{t_A}^{t_B} v(t)dt$ indicates integration of $v(t)$ from t_A to t_B with respect to t.

Integration is a type of summation, and the symbol Σ or $\int$ indicates that summation or integration.

$$\text{So, } s = \int_{t_A}^{t_B} v(t)dt \quad \ldots \quad \ldots \quad \ldots \quad (2.32)$$

$v(t)$ is called **integrand**. dt after $v(t)$ indicates that integration is to be done with respect to t.

Differentiation and integration are opposite to each other. For example,

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\text{Again, } \int \cos x \, dx = \sin x$$

That means, if $\sin x$ is differentiated we get $\cos x$, on the other hand if $\cos x$ is integrated we get $\sin x$.

2·11 Uses of vector operators

Gradient

Before defining and explaining gradient we need to know about differential operator ($\vec{\nabla}$), scalar and vector field and line integral.

Vector differential operator ∇ : Sir Hamilton first introduced the vector differential operator. Gibbs gave its name as 'del'. It's other name is nabla. Vector differential operator may be expressed in the following way :

$$\text{del, } \vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

It has vector properties like ordinary vector. It acts on a quantity and creates a new quantity.

But, $\vec{\nabla}.\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$

$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is a scalar quantity. Here $\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ means partial differential.

This operator is used to define the gradient of a scalar function, divergence of a vector function or curl of a vector function.

Scalar Field and Vector Field

Scalar field : Whatever field is considered, **physical property** is associated with each point of that field. **If the physical property associated with the field is scalar, then that field is called scalar field.** Density, temperature, potential etc are examples of scalar field. Mathematically, $\varphi(x, y, z) = 3x^2yx + 2xy^2x + 5zy^2$ indicates scalar field.

Vector field : **If the physical property associated with the field is a vector then that field is called vector field.** Velocity, electric field, gravitational field etc are examples of vector field.

$\vec{F}(x, y, z) = ax^2y\,\hat{i} + bx^2yz^2\,\hat{j} + 4zx^2\,\hat{k}$ indicates vector field.

Line integral : If $V(x, y, z)$ is a vector field, then line integral in a closed path is $\oint \vec{V}.d\vec{l}$; here $d\vec{l}$ is an element of the closed path and its direction is along the tangent of that element.

Gradient of a scalar field

Let $\varphi(x, y, z)$ indicate a differentiable scalar field. Then gradient of φ is expressed as $\vec{\nabla}\varphi$.

i.e., $\text{grad}\,\varphi = \vec{\nabla}\varphi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\varphi = \frac{\hat{i}\,\partial\varphi}{\partial x} + \frac{\hat{j}\,\partial\varphi}{\partial y} + \frac{\hat{k}\,\partial\varphi}{\partial z}$

Definition : Gradient is a vector field which expresses the maximum rate of increase of a scalar quantity. It is also called scalar function.

Physical significances of Gradient

(i) Gradient of a scalar quantity is a vector quantity.

(ii) Magnitude of that vector quantity is equal to the maximum rate of change of that scalar quantity.

(iii) Change of scalar quantity does not depend only on the coordinate of the point, but also on the direction along which the change is shown.

Mathematical example

1. If $\vec{A} = 2x^2\hat{i} + 3yz\hat{j} - xz^2\hat{k}$ and $\varphi = 2z - x^3y$, then find $\vec{A} \cdot \nabla\varphi$ at point $(1, 1, -1)$.

We know,

$$\vec{\nabla}\varphi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(2z - x^3y)$$

$$= -3x^2y\hat{i} - x^3\hat{j} + 2\hat{k}$$

Now, $\vec{A} \cdot \vec{\nabla}\varphi = (2x^2\hat{i} + 3yz\hat{j} - xz^2\hat{k}) \cdot (-3x^2y\hat{i} - x^3\hat{j} + 2\hat{k})$

$$= -6x^4y - 3x^3yz - 2xz^2$$

At point $(1, 1, -1)$,

$$\vec{A} \cdot \vec{\nabla}\varphi = -6(1)^4(1) - 3(1)^3(1)(-1) - 2(1)(-1)^2$$

$$= -6 + 3 - 2 = -5.$$

Divergence

Let in three dimensional space, the position vector of a vector field of the region R be

$\vec{V}(x, y, z) = v_1(x, y, z)\hat{i} + v_2(x, y, z)\hat{j} + v_3(x, y, z)\hat{k}$, then the scalar product of $\vec{V}$ with the operator ∇ is called divergence of that vector field. Divergence is expressed as $\vec{\nabla} \cdot \vec{V}$ or div. $\vec{V}$. Mathematically it can be written as,

$$\vec{\nabla} \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$, it is a scalar quantity.

Definition : Divergence of vector function or field is a scalar function or field which nature (external/internal) of flux at a point can be known.

Physical properties of divergence

(i) Divergence expresses the total amount of flux of a vector quantity that converges towards and diverges from a point. Divergence expresses the amount of flux. By $\vec{\nabla} \cdot \vec{V}$ or div. $\vec{V}$ it means change of density of a fluid in unit time.

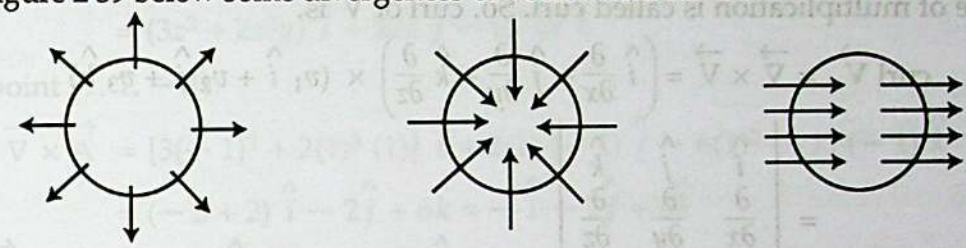
(ii) If the value is positive, then value of the fluid increases; density decreases. That means $\vec{\nabla} \cdot \vec{V}$ = '+' ve [Fig. 2·39(a)]

(iii) If the value is negative, the volume shrinks, density increases. That means $\vec{\nabla} \cdot \vec{V}$ = '–' ve [Fig. 2·39(b)]

(iv) If the value is zero, then incoming and outgoing flux becomes equal. That means $\vec{\nabla} \cdot \vec{V} = 0$ [Fig. 2·39(c)]

(v) If the divergence of a quantity becomes zero, i.e., $\vec{\nabla} \cdot \vec{V} = 0$, then that vector field is called solenoidal.

In figure 2·39 below some divergences of vector field have been shown.



(a) positive divergence (b) negative divergence (c) zero divergence

Fig. 2·39

Mathematical examples

1. Find the divergence of $\vec{A} = 3xyz^3\,\hat{i} + 2xy^2\,\hat{j} - x^3y^2z\,\hat{k}$ at position $(1, -1, 1)$.

We get,

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot (3xyz^3\,\hat{i} + 2xy^2\,\hat{j} - x^3y^2z\,\hat{k})$$

$$= 3xyz^3 + 2xy^2 - x^3y^2z$$

$$= \frac{\partial}{\partial x}(3xyz^3) + \frac{\partial}{\partial y}(2xy^2) + \frac{\partial}{\partial z}(-x^3y^2z)$$

$$= 3yz^3 + 4xy - x^3y^2$$

At position $(1, -1, 1)$

$$\vec{\nabla} \cdot \vec{A} = 3(-1)(1)^3 + 4(1)(-1) - (1)^3(-1)^2$$

$$= -3 - 4 - 1 = -8$$

2. For what value of p, $\vec{r} = (x + 3y)\hat{i} + (py - z)\hat{j} + \hat{k}(x - 2z)$ will be a solenoid?

We know, if $\vec{\nabla} \cdot \vec{r} = 0$ then $\vec{r}$ will be a solenoid.

Now, $\vec{\nabla} \cdot \vec{r} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot [(x + 3y)\,\hat{i} + (py - z)\hat{j} + \hat{k}\,(x - 2z)]$

$$= \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(py - z) + \frac{\partial}{\partial z}(x - 2z)$$

$$= 1 + p - 2 = p - 1$$

$\therefore\ \vec{\nabla} \cdot \vec{r} = p - 1 = 0$

or, $p = 1$

Curl

Let the appropriate vector function of a point in two dimensional space be $\hat{V}(x, y, z) = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$.

Then, by the vector product of ∇ and $\vec{V}$ we get a vector along the rotational axis. This type of multiplication is called curl. So, curl of $\vec{V}$ is,

$$\text{curl } \vec{V} = \vec{\nabla} \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times (v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Definition : Curl of a vector field is a vector quantity which is connected to the rotation of that field. If the magnitude of line integral per unit area around a point in the vector field becomes maximum then that expresses the curl of the vector field at that point.

Physical significances of curl

(i) Curl is a vector quantity. Its magnitude is equal to maximum line integral for unit field in that vector field. (Definition of line integral is given in article 2·12).

(ii) Direction of the vector acts along the normal on that field.

(iii) Magnitude of the vector obtained by the curl is equal to twice the angular velocity with respect to the rotation axis. That means, if

$$\vec{V} = \vec{\omega} \times \vec{r}, \text{ then } |\vec{\nabla} \times \vec{V}| = 2\vec{\omega}, \text{ here } \vec{\omega} \text{ is a constant vector.}$$

(iv) Curl of a vector indicates the rotation of that vector. Curl represents the number of rotation of the vector around a point.

(v) Gradient of curl of any vector field is zero. That means $\vec{\nabla} . (\vec{\nabla} \times \vec{V}) = 0$.

(vi) If the curl of a vector is zero, then the vector is irrotational.

Mathematical examples

1. Find the curl of $\vec{A} = xz^2 \hat{i} - 2x^3 yz \hat{j} + 3yz^3 \hat{k}$ at point (1, 1, − 1).

We get,

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2 & -2x^3yz & 3yz^3 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3yz^3) + \frac{\partial}{\partial z}(2x^3yz)\right]\hat{i} + \left[\frac{\partial}{\partial z}(xz^2) - \frac{\partial}{\partial x}(3yz^3)\right]\hat{j} + \left[\frac{\partial}{\partial x}(-2x^3yz) - \frac{\partial}{\partial y}(xz^2)\right]\hat{k}$$

$$= (3z^3 + 2x^3y)\,\hat{i} + (2xz)\,\hat{j} + (-6x^2yz)\,\hat{k}$$

$$= (3z^3 + 2x^3y)\,\hat{i} + 2xz\,\hat{j} - 6x^2yz\,\hat{k}$$

At point (1, 1, − 1),

$$\vec{\nabla} \times \vec{A} = [3(-1)^3 + 2(1)^3(1)]\,\hat{i} + 2(1)(-1)\,\hat{j} - 6(1)^2(+1)(-1)\,\hat{k}$$

$$= (-3 + 2)\,\hat{i} - 2\hat{j} + 6\hat{k} = -\hat{i} - 2\hat{j} + 6\hat{k}$$

2. $\vec{A} = (6xy + z^3)\,\hat{i} + (3x^2 - z)\,\hat{j} + (3xz^2 - y)\,\hat{k}$**, prove that the vector** $\vec{A}$ **is irrotational**

We know, vector $\vec{A}$ will be irrotational, if $\vec{\nabla} \times \vec{A} = 0$.

Now

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (6xy + z^3) & (3x^2 - z) & (3xz^2 - y) \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(3xz^2 - y) - \frac{\partial}{\partial z}(3x^2 - z)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(3xz^2 - y) - \frac{\partial}{\partial z}(6xy + z^3)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(3x^2 - z) - \frac{\partial}{\partial y}(6xy + z^3)\right\}$$

$$= \hat{i}\,\{(0 - 1) - (0 - 1)\} - \hat{j}\,\{(3z^2 - 0) - (0 + 3z^2)\} + \hat{k}\,\{(6x - 0) - (6x + 0)\}$$

$$= \hat{i}\,(-1 + 1) - \hat{j}\,(3z^2 - 3z^2) + \hat{k}\,(6x - 6x)$$

$$= 0$$

$$\therefore\ \vec{\nabla} \times \vec{A} = 0$$

Hence, the vector $\vec{A}$ is irrotational.

Necessary mathematical formulae

$\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$ ... ... (1)

$\hat{i}.\hat{j} = \hat{i}.\hat{k} = \hat{j}.\hat{i} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = \hat{k}.\hat{j} = 0$ ... ... (2)

$\hat{i} \times \hat{j} \times \hat{k} = 0$ ... ... (3)

$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i}$ ... ... (4)

$\hat{k} \times \hat{i} = \hat{j}$ ... ... (5)

$\hat{i} \times \hat{k} = -\hat{j}$ ... ... (6)

$\hat{j} \times \hat{i} = -\hat{k}$ ... ... (7)

$\hat{k} \times \hat{j} = -\hat{i}$ ... ... (8)

$R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$ ... ... (9)

$\tan\theta = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$ ... ... (10)

$R_{max} = P + Q$ ... ... (11)

$R_{min} = P \sim Q$ ... ... (12)

$\vec{A}.\vec{B} = 0$ (condition for normal) ... ... (13)

$\vec{A} \times \vec{B} = 0$ (condition for parallel) ... ... (14)

$\vec{A}.\vec{B} = AB\cos\theta$ ... ... (15)

$\vec{A} \times \vec{B} = \hat{n}AB\sin\theta$ ... ... (16)

$|\vec{A} \times \vec{B}| = AB\sin\theta$ ... ... (17)

unit vector, $\hat{n} = \dfrac{\vec{A}}{|\vec{A}|}$ ... ... (18)

magnitude of a vector, $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$ ... ... (19)

normal unit vector, $\hat{n} = \dfrac{|\vec{A} \times \vec{B}|}{|\vec{A} \times \vec{B}|}$ ... ... (20)

perpendicular projection along A, $B\cos\theta = \dfrac{\vec{A}.\vec{B}}{|\vec{A}|}$ ... ... (21)

work, $W = \vec{F}.\vec{s}$ ... ... (22)

torque, $\vec{\tau} = \vec{P} \times \vec{r}$ ... ... (23)

area of a parallelogram $= |\vec{A} \times \vec{B}|$ ... ... (24)

area of a triangle $= \frac{1}{2} \times$ base $\times$ height ... ... (25)

$\vec{A}.(\vec{B} \times \vec{C}) = 0$ (condition of three vectors to become coplanar) ... (26)

resolution of $\vec{F}$ in components, $F_x = F\cos\theta$ and $F_y = F\sin\theta$ ... (27)

$\vec{\nabla}.\vec{V} = 0$ (condition of solenoid) ... ... (28)

$\vec{\nabla} \times \vec{V} = 0$ (condition for irrotation) ... ... (29)

Higher efficiency mathematical examples

1. Tamal is going to school from home by cycle. All on a sudden, it starts raining. Drops of rain start falling on him at $6\ ms^{-1}$. Flow of air was not significant. Even then, drops of rain fall on him at an angle of 45°.

(a) Calculate the velocity at the cycle.

(b) If Tamal drives his cycle at double speed in order to reach school quickly what angle he will have to hold an umbrella to protect himself from rain ?

(a) We know,

$$\tan\theta = \frac{Q\sin\alpha}{P + Q\cos\alpha}, \text{ here } \alpha = 90°$$

$$\text{or,}\quad \tan\theta = \frac{Q\sin 90°}{P + Q\cos 90°}$$

$$\text{or,}\quad \tan 45° = \frac{6\sin 90°}{P + Q\cos 90°} = \frac{6}{P}$$

$$\therefore\quad P = 6\ ms^{-1}$$

$\therefore$ Velocity of the cycle is $6\ ms^{-1}$

Here,

velocity of rain, $Q = 6\ ms^{-1}$

velocity of cycle = P

$\theta = 45°$

$\alpha = 90°$

(b) If the cycle is driven at double speed, then changed velocity,

$$P' = (6 \times 2) = 12\ ms^{-1}$$

$$\text{or, } \tan\theta = \frac{6\sin 90°}{12 + 6\cos 90°} = \frac{6}{12 + 0} = \frac{1}{2}$$

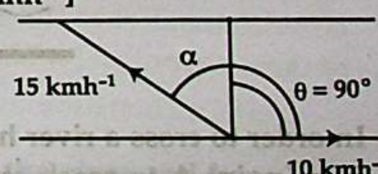
$$\therefore\quad \theta = 26^{\cdot}57°$$

That is, he has to hold the umbrella at 26·57° angle.

2. In order to go directly to the opposite side of the river by a ferry that is moving at a velocity of $15\ kmh^{-1}$, Swapan noticed that the ferry, instead of going straight was moving obliquely. [Velocity of current $=10\ kmh^{-1}$]

(a) How many times is the maximum value of the resultant to that of the minimum.

(b) Analyse the reason for changing the direction of the ferry.



(a) We know,

$$R_{max} = v + u = 15 + 10 = 25\ kmh^{-1}$$

$$\text{Again,}\quad R_{min} = v - u = 15 - 10 = 5\ kmh^{-1}$$

$$\therefore\quad \frac{R_{max}}{R_{min}} = \frac{25}{5} = 5$$

$$\Rightarrow R_{max} = 5\,R_{min}$$

Here,

Velocity of the ferry, $v = 15\ kmh^{-1}$

Velocity of current, $u = 10\ kmh^{-1}$

$\frac{R_{max}}{R_{min}} = ?$

i.e., Maximum value of the resultant is 5 times the minimum value of the resultant.

(b) Since there is current in the river, hence the ferry is to start obliquely in order to reach directly opposite to the river.

Let the ferry need to start against the current at an angle α.

In that case, velocity of current, $u = 10 \text{ kmh}^{-1}$, velocity of the ferry, $v = 15 \text{ kmh}^{-1}$ and the direction of the resultant velocity along the direction of current will be $\theta = 90°$.

We know, $\tan\theta = \dfrac{v\sin\alpha}{u + v\cos\alpha}$

or, $\tan 90° = \dfrac{v\sin\alpha}{u + v\cos\alpha}$

or, $\dfrac{1}{0} = \dfrac{v\sin\alpha}{u + v\cos\alpha}$

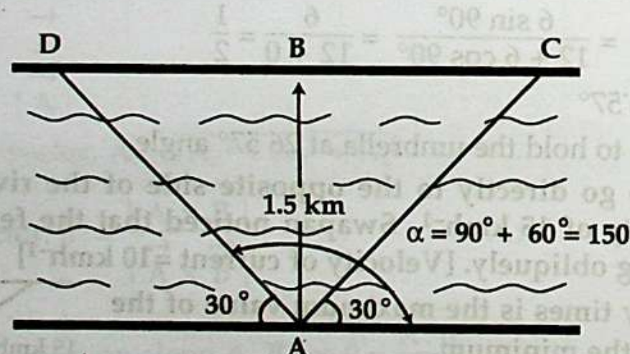
or, $u + v\cos\alpha = 0$

or, $\cos\alpha = -\dfrac{u}{v} = -\dfrac{10}{15}$

or, $\alpha = 131.8°$

Since, $\alpha > 90°$, so the ferry had to start obliquely instead of starting directly.

3.



In order to cross a river having current whose width is 1.5 km Pintu decided to start from point A to reach straight to the point B on the opposite side of the river by swimming. The velocity of current in that river was 3 kmh⁻¹ and the velocity of the swimmer was 4 kmh⁻¹. Due to current Pintu although started along AB but reached the opposite side along the line AC.

(a) Calculate the distance travelled along AC.

(b) By swimming along AD could Pintu reach the point B ? Give your opinion with mathematical analysis.

(a) In the stimulus, AB = 1.5 km, by resolving AB along AC, we get,

$$AC \sin 30° = AB$$

$$\therefore \text{ distance travelled, } AC = \frac{AB}{\sin 30°} = \frac{1.5}{0.5} = 3 \text{ km}$$

(b) Suppose, by starting against the current at an angle α Pintu would reach at point B. Here velocity of the current, $u = 3$ kmh^{-1} and velocity of the swimmer, $v = 4$ kmh^{-1} and the direction of the resultant velocity with the current = 90°.

We know,

$$\tan 90° = \frac{v \sin \alpha}{u + v \cos \alpha}$$

or, $\frac{1}{0} = \frac{v \sin \alpha}{u + v \cos \alpha}$

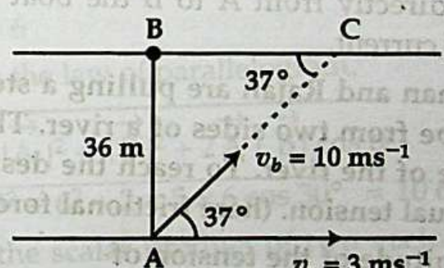
or, $u + v \cos \alpha = 0$

or, $v \cos \alpha = -u$

or, $\alpha = \cos^{-1}\left(-\frac{u}{v}\right) = \cos^{-1}\left(-\frac{3}{4}\right) = 138^{\cdot}59°$

Since the angle subtended along AD by the current was 150°; so by swimming along AD Pintu would not be able to reach the point B. Hence Pintu would have to reach the point B by swimming from the point A making an angle of 138·59° with the current.

4. A boat is moving with velocity 10 ms^{-1} in a river of width 36 m. By crossing the river the boat reached at point C in the opposite side. Velocity of current in the river is 3 ms^{-1}. [C. B. 2015]



(a) Calculate the distance BC on the opposite side of the river and how long will it take to cross the river ?

(b) What measure is to be taken by the boatman to reach the boat directly to the point B in the opposite side of the river ?

(a) component of velocity along the width of the river = $v_b \sin 37° = 10 \sin 37° = 6^{\cdot}02$ ms^{-1}

time taken to cross the river, $t = \frac{d}{6^{\cdot}02} = \frac{36}{6^{\cdot}02} = 5^{\cdot}982$ sec

According to the figure, $\sin 37° = \frac{AB}{AC} = \frac{36}{AC}$

or, $AC = \frac{36}{\sin 37°} = 59^{\cdot}81$ m

From the right angled triangle, ABC we get,

$$AC^2 = AB^2 + BC^2$$

or, $BC^2 = AC^2 - AB^2 = (59{\cdot}81)^2 - (36)^2$

or, $BC^2 = 2282{\cdot}318$

$\therefore$ $BC = 47{\cdot}77$ m

(b) In order to reach the boat directly to the opposite side at point B, angle between the resultant velocity of the boat and current and the velocity of current should be $\theta = 90°$. If the angle between the boat and current is α, then

We get,

$$\tan 90° = \frac{v_b \sin \alpha}{v_r + v_b \cos \alpha}$$

or, $\frac{1}{0} = \frac{v_b \sin \alpha}{v_r + v_b \cos \alpha}$

or, $v_r + v_b \cos \alpha = 0$

or, $\cos \alpha = \frac{-v_r}{v_b}$

$\therefore$ $\alpha = \cos^{-1}\left(\frac{-v_r}{v_b}\right) = \cos^{-1}\left(-\frac{3}{10}\right)$

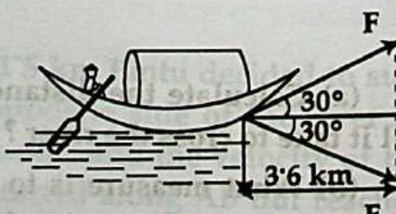
$= \cos^{-1}(-0{\cdot}3) = 107{\cdot}45°$

So, in order to reach directly from A to B the boat is to travel at an angle of 107·45° along the direction of current.

5. In steady water Kanan and Rajan are pulling a steady boat of weighing 500 kg at an angle of 30° by rope from two sides of a river. The boat travels 3·6 km in 5 minutes parallel to the side of the river. To reach the destination in 5 minutes both of them are pulling with equal tension. (here frictional force is negligible)

(a) Calculate the magnitude of the tension of rope.

(b) Is it posible to reach to the destination in the time mentioned in the stimulus ? Show by mathematical analysis.



(a) If the tension is F, then the summation of the two components of the force,

$$F \cos \theta + F \cos \theta = ma$$

or, $2F \cos \theta = m \times \frac{2s}{t^2}$

$\therefore$ $F = \frac{m \times 2s}{t^2 \times 2 \cos \theta}$

$= \frac{500 \times 2 \times 3{\cdot}6 \times 10^3}{(300)^2 \times 2 \cos 60°} = 23{\cdot}09$ N

here,
$\theta = 30°$
distance, $s = 3{\cdot}6 \times 10^3$ m
time, $t = 5 \times 60 = 300$ s
mass of the boat, $m = 500$ kg
tension, F = ?

(b) If the tension are equal, then total tension,

$$F + F = ma$$

or, $2F = ma = m \times \frac{2s}{t^2}$

or, $t^2 = \frac{2 \times 500 \times 3.6 \times 10^3}{2 \times 23.09} = 77955.825$

$\therefore \quad t = 279.20 \text{ sec} = 4.65 \text{ min}$

here $t < 5$ min

So, they will reach in time.

here,

$s = vt + \frac{1}{2}at^2$

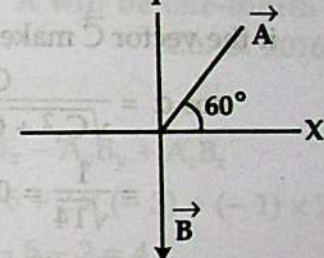
or, $s = \frac{1}{2}at^2 \quad [\because v = 0]$

$\therefore \; a = \frac{2s}{t^2}$

6. In the figure $|\vec{A}| = 5$ and $|\vec{B}| = 6$

(a) What is the magnitude of $(\vec{A} - \vec{B})$?

(b) The vector $(\vec{A} \times \vec{B})$ is perpendicular to $(\vec{A} + \vec{B})$ —verify with mathematical analysis. [D. B. 2016]



(a) We know,

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

here, angle between $\vec{A}$ and $\vec{B}$, $\alpha = 90 - 60° = 30°$

$|\vec{A}| = 5, |\vec{B}| = 6$

Now, according to the law of parallelogram,

$$|\vec{A} - \vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A}\,|\,\vec{B}|\cos\alpha}$$

$$= \sqrt{5^2 + 6^2 + 2 \times 5 \times 6 \cos 30°} = 10.63$$

(b) We know, if the scalar product of two vectors is zero, then the two vectors become mutually perpendicular.

So, scalar product of $(\vec{A} \times \vec{B})$ and $(\vec{A} + \vec{B})$ i.e., if $(\vec{A} \times \vec{B}) \cdot (\vec{A} + \vec{B})$ is zero, then the vector $(\vec{A} \times \vec{B})$ will be perpendicular to $(\vec{A} + \vec{B})$.

$$\vec{A} + \vec{B} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} + B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

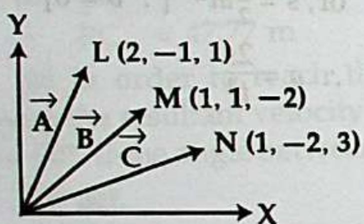
$$= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} + (A_z + B_z)\hat{k}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}(A_yB_z - A_zB_y) + \hat{j}(A_zB_x - A_xB_z) + \hat{k}(A_xB_y - A_yB_x)$$

Now, $(\vec{A} \times \vec{B}) \cdot (\vec{A} + \vec{B}) = (A_yB_z - A_zB_y)(A_x + B_x) + (A_zB_x - A_xB_z)(A_y + B_y)$
$+ (A_xB_y - A_yB_x)(A_x + B_z) = 0$ (Proved)

7.



(a) What is the angle of $\vec{C}$ along the X-axis ?

(b) Verify mathematically whether the vector perpendicular to the two vectors $\vec{B}$ and $\vec{C}$ is in the same plane with vector $\vec{A}$ or not. [C. B. 2016]

(a) We know,

if the vector $\vec{C}$ makes an angle α with X-axis, then,

here, $\vec{C} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\cos\alpha = \frac{C_x}{\sqrt{C_x^2 + C_y^2 + C_z^2}} = \frac{1}{\sqrt{1+4+9}}$$

$$= \frac{1}{\sqrt{14}} = 0{\cdot}26726$$

$\therefore\ \cos\alpha = 0{\cdot}26726$

$\therefore\ \alpha = \cos^{-1}(0{\cdot}26726) = 74{\cdot}5°$

(b) We know, vector product of two vectors is a vector quantity whose direction is perpendicular to the direction of two vectors.

Let $\vec{D} = \vec{B} \times \vec{C}$

here, $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{B} = \hat{i} + \hat{j} - 2\hat{k}$, $\vec{C} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\vec{D} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -2 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= \hat{i}\,(3-4) + \hat{j}\,(-2-3) + \hat{k}\,(-2-1)$$

$$= -\hat{i} - 5\hat{j} - 3\hat{k}$$

Now, $\vec{D}$ and $\vec{A}$ will remain in the same plane if the two vectors are mutually perpendicular.

That means, their scalar product is zero.

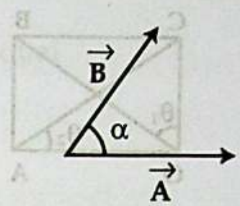
Now, $\vec{D} \cdot \vec{A} = D_xA_x + D_yA_y + D_zA_z$

$= (-1) \times 2 + (-5) \times (-1) + (-3) \times 1$

$= -2 + 5 - 3 = 0$

Since $\vec{D}$ and $\vec{A}$ are mutually perpendicular, so they must remain in the same plane.

8.



$$\vec{A} = 2\hat{i} + 2\hat{j} - \hat{k}$$
$$\vec{B} = 6\hat{i} - 3\hat{j} + 2\hat{k}$$

(a) What is the value of α ?

(b) For what change in the value of α projection of $\vec{B}$ on $\vec{A}$ will be one-fourth ? Give opinion with mathematical analysis. [Ch. B. 2016]

(a) We know,

$\vec{A} \cdot \vec{B} = AB \cos \alpha$

$\therefore \cos \alpha = \dfrac{\vec{A} \cdot \vec{B}}{AB}$ ... ... (i)

here,

$\vec{A} \cdot \vec{B} = A_xB_x - A_yB_y + A_zB_z$
$= 2 \times 6 + 2 \times (-3) + (-1) \times 2$
$= 12 - 6 - 2 = 4$

$A = |\vec{A}| = \sqrt{4+4+1} = 3$

$B = |\vec{B}| = \sqrt{36+9+4} = 7$

$\therefore \cos \alpha = \dfrac{4}{3 \times 7} = 0{\cdot}19048$

$\alpha = \cos^{-1} (0{\cdot}19048) = 79{\cdot}02°$

(b) In the first case, projection of $\vec{B}$ on $\vec{A}$,

$B_1 = B \cos \alpha = 7 \cos 79{\cdot}02° = 1{\cdot}33$

here,
$A = 3$
$B = 7$
$\alpha = 79{\cdot}02°$

In the 2nd case, if projection is B_2,

$B_2 = \dfrac{1}{4} B_1 = \dfrac{1}{4} \times 1{\cdot}33$

$= 0{\cdot}3325$

Again, if the angle between $\vec{A}$ and $\vec{B}$ is β.

projection, $B \cos \beta = B_2$

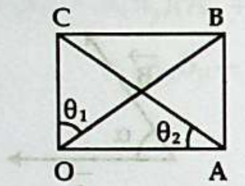
$\therefore \cos \beta = \dfrac{B_2}{B} = \dfrac{0{\cdot}3325}{7} = 0{\cdot}0475$

$\therefore \beta = \cos^{-1} (0{\cdot}0475) = 87{\cdot}28°$

$\therefore$ the angle is to be increased by

$87{\cdot}28° - 79{\cdot}02° = 8{\cdot}26°$

9.



OABC in the above figure is a rectangle. Two vectors $\vec{P} = \hat{i} - 2\hat{j} - \hat{k}$ and $\vec{Q} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ are respectively represented by arms OA and OB.

(a) Calculate the area of ΔOAB according to the stimulus.

(b) Find out by mathematical analysis which of the angles of θ_1 and θ_2 according to the stimulus is greater ? [Ch. B. 2017]

(a) Given, vector represented by the arm OA, $\vec{P} = \hat{i} - 2\hat{j} - \hat{k}$

and vector represented by the arm OB, $\vec{Q} = 2\hat{i} - 3\hat{j} + 2\hat{k}$

Now, $\vec{P} \times \vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -1 \\ 2 & -3 & 2 \end{vmatrix}$

$= \hat{i}\,(-4-3) - \hat{j}\,(2+2) + \hat{k}\,(-3+4)$

$= -7\hat{i} - 4\hat{j} + \hat{k}$

$\therefore \; |\vec{P} \times \vec{Q}| = \sqrt{(-7)^2 + (-4)^2 + (1)^2} = \sqrt{66}$

$\therefore$ area of $\Delta AOB = \frac{1}{2} \times |\vec{P} \times \vec{Q}| = \frac{1}{2} \times \sqrt{66} = 4.062$ sq. unit

(b) From the above (a) we get,

$|\vec{P} \times \vec{Q}| = \sqrt{66}$

or, $PQ \sin\theta = \sqrt{66}$

or, $(\sqrt{(1)^2 + (-2)^2 + (-1)^2} \times \sqrt{(2)^2 + (-3)^2 + (2)^2}) \sin\theta = \sqrt{66}$

or, $(\sqrt{6} \times \sqrt{17}) \sin\theta = \sqrt{66}$

or, $\sqrt{23} \sin\theta = \sqrt{66}$

or, $\sin\theta = \frac{\sqrt{66}}{\sqrt{23}}$

$\therefore \; \theta = \sin^{-1}\left(\frac{\sqrt{66}}{\sqrt{23}}\right) = 53.55^\circ$

So, angle between OA and OB, $\theta = 53.55^\circ$

$\therefore$ OABC is a rectangle and

$\angle AOC = 90^\circ$

$\therefore \quad \theta_1 = 90° - \theta = 90° - 53'55°$

$\boxed{\theta_1 = 36'45°}$

Again, ΔAOC and ΔOAB are equal

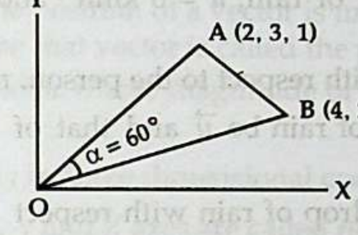
So, $\angle AOB = \angle OAC$

$\therefore \quad \theta_2 = \theta$

$\therefore \quad \boxed{\theta_2 = 53'55°}$

Since, $\theta_2 > \theta_1$, so angle θ_2 is greater than angle θ_1.

10. In the following figure co-ordinates of two points A and B are given—



(a) Calculate the magnitude of the vector connecting AB.

(b) Will the triangle in the stimulus form equiangular triangle ? Give opinion analytically. [B. B. 2017]

(a) Here,

$\vec{OA} = 2\hat{i} + 3\hat{j} + \hat{k}$

and $\vec{OB} = 4\hat{i} + 2\hat{j} - 3\hat{k}$

Given,

co-ordinate of A (2, 3, 1)

co-ordinate of B (4, 2, –3)

vector $\vec{AB}$ connecting AB = ?

We know, $\vec{AB} = \vec{OB} - \vec{OA}$

$= (4\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} + 3\hat{j} + \hat{k})$

$= 2\hat{i} - \hat{j} - 4\hat{k}$

So, magnitude of the vector connecting AB = $|\vec{AB}| = \sqrt{(2)^2 + (-1)^2 + (-4)^2} = \sqrt{21}$

(b) From above (a) we get,

$|\vec{AB}| = \sqrt{(2)^2 + (-1)^2 + (-4)^2} = \sqrt{21}$

$|\vec{OA}| = \sqrt{(2)^2 + (3)^2 + (1)^2} = \sqrt{14}$

$|\vec{OB}| = \sqrt{(4)^2 + (2)^2 + (-3)^2} = \sqrt{29}$

Given,

$\vec{AB} = 2\hat{i} - \hat{j} - 4\hat{k}$

$\vec{OA} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{OB} = 4\hat{i} + 2\hat{j} - 3\hat{k}$

Here, $|\vec{OB}|^2 = \sqrt{(29)^2} = 29$

$|\vec{OA}|^2 + |\vec{AB}|^2 = (\sqrt{(14)^2} + (\sqrt{(21)^2} = 35$

i.e., $|\vec{OB}|^2 \neq |\vec{OA}|^2 + |\vec{AB}|^2$

So, the triangle of the stimulus will not form equianglular triangle.

11. In a rainy day Nafisa standing besides the window observed that rain was pouring vertically at velocity of 6 kmh^{-1}. Nafisa noticed that in the street a man was walking at velocity of 4kmh^{-1} and another person was moving in a cycle at velocity of 8 kmh^{-1}. Their umbrellas were held at different angles.

(a) What is the resultant velocity of falling rain water with respect to the walking person of the stimulus ?

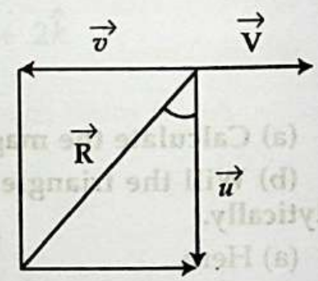
(b) Analyse mathematically the observation of Nafisa that the umbrellas of the walking person and the person in the cycle were not inclined in the same way.

[R. B. 2017]

(a) Suppose, the velocity of rain, u = 6 kmh^{-1} and the velocity of the walking person, v = 4 kmh^{-1}

Relative velocity of rain with respect to the person, v_r = ?

Let, velocity of the drop of rain be $\vec{u}$ and that of the cycle be $\vec{v}$.



So, if the velocity of the drop of rain with respect to the cycle is v_r, then

$$v_r = \sqrt{u^2 + v^2 + 2uv \cos 90°}$$
$$= \sqrt{(6)^2 + (4)^2 + 2(6)(4) \cos 90°}$$
$$= \sqrt{36 + 16 + 0}$$
$$= \sqrt{52} = 7{\cdot}21 \text{ kmh}^{-1}$$

Again, $\tan\theta = \dfrac{v \sin 90°}{u + v \cos 90°} = \dfrac{4 \sin 90°}{6 + 4 \cos 90°} = \dfrac{4}{6} = 0{\cdot}666667$

$\therefore\ \theta = 33{\cdot}69°$

So, relative velocity of rain with respect to the person is 7·21 km^{-1}. This velocity makes an angle of 33·69° with the vertical.

(b) According to the stimulus,

velocity of the walking person, v_1 = 4 kmh^{-1}
velocity of the person in the running cycle, v_2 = 8 kmh^{-1}
velocity of rain, u = 6 kmh^{-1}

Suppose the person walking is to hold the umbrella at angle of θ_1 with the vertical and the person in the cycle is to hold the umbrella at an angle of θ_2 with the vertical in order to protect him from the rain.

$\therefore\ \tan\theta_1 = \dfrac{v_1 \sin 90°}{6 + 4 \cos 90°} = \dfrac{4}{6} = 0{\cdot}666667$

$\therefore\ \theta_1 = 33{\cdot}69°$

and $\tan\theta_2 = \dfrac{8 \sin 90°}{6 + 8 \cos 90°} = \dfrac{8}{6} = 1{\cdot}33333$

$\therefore\ \theta_2 = \tan^{-1}\left(\dfrac{8}{6}\right) = 53{\cdot}13°$

Summary

Vector quantity : A physical quantity which has both magnitude and direction is called vector quantity or vector.

Scalar quantity : A physical quantity which has only magnitude but no direction is called scalar quantity or scalar.

Unit vector : A vector of unit magnitude is called a unit vector.

Resultant and components : The addition of two or more vectors is called resultant and each of these vectors is called component.

Position vector : When the position of a vector is uniquely specified with reference to the origin of a reference frame that vector is called the position vector.

Null or zero vector : A vector whose magnitude or modulus is zero is called a null or zero vector.

Rectangular unit vectors : In three dimensional coordinate system, unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ along the respective X, Y and Z axes are called rectangular unit vectors.

Displacement vector : Distance travelled by a particle in linear or straight path or in a particular direction is called displacement vector.

Radius vector : The distance from the origin to a point is called radius vector.

Like vectors : Two vectors of same type but of different magnitude parallel to each other and directed along the same direction are called like vectors.

Reciprocal vectors : If the magnitude of one of the two parallel vectors is reciprocal of the other vector, then they are called reciprocal vectors.

Collinear vectors : If two or more vectors are directed along the same line or parallel to one another, then the vectors are called collinear vectors.

Coplanar vectors : If two or more vectors lie on the same plane, then the vectors are called coplanar vectors.

Scalar field : If the physical properties associated with a field is scalar, then that field is called scalar field.

Vector field : If the physical property associated with the field is a vector, then that field is called vector field.

Resolution of a vector and components : The process of resolving a vector into two or more vectors is called resolution of a vector and each resolving vector is called component of the original vector.

Triangle law : If two similar vectors acting at a point can be represented by two consecutive sides of a triangle taken in order, then the third side will give the resultant vector in the reverse order.

Parallelogram law of addition : If two similar vectors acting simultaneously at a point can be represented both in magnitude and direction by two adjacent sides of a parallelogram, then the diagonal from the point of intersection of these sides give the resultant vector both in magnitude and direction. It is called the parallelogram law of vector addition.

Scalar or dot product : It is defined as the product of the magnitudes of the two vectors $\vec{A}$ and $\vec{B}$ and the cosine of their included angle.

Vector product or cross product : If the product of two vectors is a vector quantity, then that product is called a vector or cross product. The magnitude of that vector is the product of the magnitude of those vectors and the cosine of their included angle. The direction of vector product is determined by the right-handed screw rule.

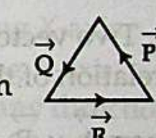
Operator : The mathematical symbol or index by which one quantity can be transformed into another quantity or can explain a variable quantity, then it is called an operator.

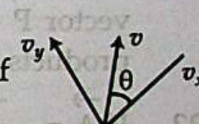
Divergence : In three dimensional space, if the position vector of a vector field in region R is $\vec{V}(x, y, z) = v_1(x, y, z)\,\hat{i} + v_2(x, y, z)\,\hat{j} + v_3(x, y, z)\,\hat{k}$, then the scalar product of the operator ∇ with $\vec{V}$ is called the divergence of that vector field.

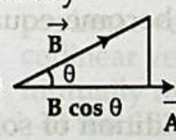
Curl : In three dimensional space, if the appropriate vector function of a point is $\vec{V}(x, y, z) = v_1\,\hat{i} + v_2\,\hat{j} + v_3\,\hat{k}$, then a vector along the rotational axis is obtained by the cross product of the operator ∇ and $\vec{V}$. This type of product is called curl.

Summary of the relavant topics for the answer of multiple choice questions

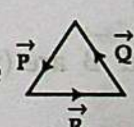
1. $\vec{A}.\vec{B} = 0$ means—
   (a) $\vec{A} = 0$ (b) $\vec{B} = 0$ (c) $\vec{A}$ and $\vec{B}$ are mutually perpendicular.
2. If $\vec{A}.(\vec{B} \times \vec{C}) = 0$ it indicates that the three vectors are coplanar. Maximum magnitude of the resultant of $\vec{A}$ and $\vec{B}$ is $\vec{A} + \vec{B}$ and minimum magnitude $\vec{A} - \vec{B}$.
3. If $\vec{C} = \vec{A} \times \vec{B}$ and $\vec{D} = \vec{B} \times \vec{A}$ then angle between $\vec{C}$ and $\vec{D}$ is 180°.
4. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then magnitude of $\vec{\nabla}.\vec{r}$ will be 3. Magnitude of $2\hat{i} + 3\hat{j}$ is $\sqrt{13}$. It is located in the XY plane making an angle of 90° with Z axis.
5. Angle between $\vec{A}$ and its unit vector $(\hat{a})$ is 0°. If $Q(x, y) = 3x^2y$, then at point $(1, -2)$ $\vec{\nabla} = -12\hat{i} + 3\hat{j}$.
6. If $\vec{A} = \hat{i}$ and $\vec{B} = \hat{j} + \hat{k}$, then angle between $\vec{A}$ and $\vec{B}$ is 90°. If $|\vec{A}.\vec{B}| = |\vec{A} \times \vec{B}|$, then angle between $\vec{A}$ and $\vec{B}$ is 45°. If angle between $\vec{F}$ and $\vec{s}$, $\theta = 90°$, then the work done is zero.
7. Magnitude of $\hat{i} \times (\hat{j} \times \hat{k})$ is zero, the value of $\hat{j} \times (\hat{j} \times \hat{k})$ is $-\hat{k}$. Normal vector on $\vec{P} = 4\hat{i} + 3\hat{j}$ is $3\hat{i} - 4\hat{j}$.

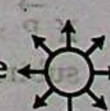
8. If two vectors act at an angle of 45°, then their scalar and vector products become equal.
9. Parallel vector of X-axis is $(\hat{i} \times \hat{j}) \times \hat{j}$. Length of the vector $3\hat{i} + 2\hat{j} + 4\hat{k}$ in YZ plane is $\sqrt{50}$.
10. Magnitude of the resultant of two vectors will be maximum when angle between them is 0°.
11. $|\vec{A} \times \vec{A}| = 0$. If $\vec{A} \times \vec{B} = 0$ then $\vec{A}$ and $\vec{B}$ are parallel to each other. $\vec{A}$ and $\vec{B}$ will be reciprocal, when $\vec{A} = 2\hat{i}$ and $\vec{B} = \frac{1}{2}\hat{i}$. Direction of the vector $\hat{i} \times \hat{j}$ is along $\hat{k}$.
12. If the diagonals of a parallelogram are $2\hat{i}$ and $2\hat{j}$, then its area will be 2 square unit. Gradient transforms a scalar function into vector quantity.
13. Normal projection of $\vec{B}$ along $\vec{A}$ is B cos θ. Angle between $\vec{A} \times \vec{B}$ and $(\vec{A} + \vec{B})$ is 90°.
14. For two equal forces and for angle 60° between them, the square of the resultant will be 3 times their magnitude.
15. $\vec{P} + \vec{Q} + \vec{R} = 0$ can be expressed by the diagram 

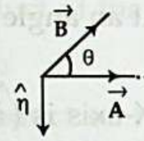
. If 

$\theta = 45°$, then components v_x and v_y will be equal. Energy and potential are scalar quantity.
16. (a) 

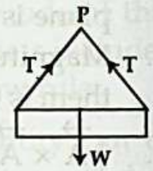
here B cos θ is normal component or projection of $\vec{B}$ along $\vec{A}$.

(b) If the initial and end points of a vector are same, then the vector will be a null vector.
17. $\vec{P} + \vec{Q} = \vec{R}$ is expersed by the figure 

. If a vector whose magnitude is not zero is divided by its magnitude, then unit vector is obtained. The figure  is the divergence of a vector field, so $\vec{\nabla} \cdot \vec{V}$ = '+'ve;  the figure is the vector divergence of a vector field, so $\vec{\nabla} \cdot \vec{V}$ = '−' ve; the figure  is zero divergence, so $\vec{\nabla} \cdot \vec{V} = 0$.

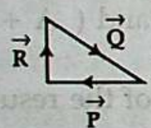
18. Figure of a normal unit vector $\hat{\eta} = \dfrac{\vec{B} \times \vec{A}}{|\vec{B} \times \vec{A}|}$ is 

.. If $|\vec{A} \times \vec{B}| = \vec{A} \cdot \vec{B}$, then angle between them is $\frac{\pi}{4}$. In the figure two ends of a rectangular frame of weight W is fastened by a thread and the mid point is attached to a wall. Then the value of W will be W = 2T cos θ.



19. Magnitude of the vector $\hat{i} + \hat{j} + \hat{k}$ will be $\sqrt{3}$. $\hat{j} \times (\hat{j} \times \hat{k}) = -\hat{k}$; If the three arms of a triangle are represented by $\vec{P}, \vec{Q}, \vec{R}$ in the same order, then $\vec{P} + \vec{Q} + \vec{R} = 0$.

20. $\vec{A} \cdot \vec{A} = A^2$, the vector $\hat{i} + \hat{j}$ is located in the XY plane. If $\vec{A} = 5\,\hat{i}$, $\vec{B} = \frac{1}{5}\,\hat{i}$, then the two vectors will be reciprocal.

21. 

In this figure vector $\vec{R}$ represents the magnitude and direction of the vector $\vec{P} - \vec{Q}$. Two vectors will be parallel when $\vec{A} \times \vec{B} = 0$. In scalar and vector products the relation of the angle between the two vectors is $0° \le \theta \le 180°$.

22. If $\vec{A} = -\vec{B}$, then $\vec{A} \times \vec{B} = 0$. Condition of vectors $\vec{A} + \vec{B}$ and $\vec{A} - \vec{B}$ to become equal is $\vec{B} = 0$.

23. Magnitudes of the addition and subtraction of two vectors become equal when angle between them is 90°.

24. $\vec{\nabla} \times \vec{V} = 0$, is the irrotational condition. $\vec{\nabla} \cdot \vec{V} = 0$ is the condition of solenoidal and then incoming and outgoing flux become equal. Gradient transforms a scalar quantity into a vector quantity.

25. If $\vec{A} = 2\vec{B}$, then vectors $\vec{A}$ and $\vec{B}$ are (i) like vector (ii) act in the same direction (iii) similar vectors.

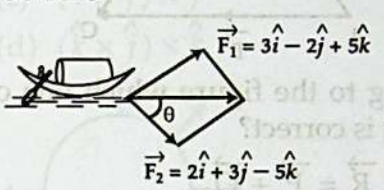
26. If $\vec{P} = \vec{Q}$, then magnitude of $\vec{P} \times (\vec{Q} \times \vec{P})$ is zero. Magnitudes of addition and subtraction of two vectors become same, when the angle between them is 90°. In figure 

the normal projection of $\vec{P}$ in the direction of $\vec{Q}$ is P cos θ.

27. Angle between $\hat{i}$ and $-\hat{i}$ is 180°; $|-\hat{i} - \hat{j} - \hat{k}| = \sqrt{3}$. Vector divergence is a scalar quantity.

28. Curl of a vector field is a vector quantity. If the curl of a vector is zero, then that is irrotational.

29. The vector $2\hat{i} + 3\hat{j}$ is in the XY plane. If $\vec{A} = -3\vec{B}$ vectors, then $\vec{A}$ and $\vec{B}$ are similar and act opposite to each other. If $\vec{A} = -\vec{B}$, then magnitude of $\vec{A} \times \vec{B}$ will be zero.

30. 

The magnitude of the resultant of the vectors $\vec{F_1}$ and $\vec{F_2}$ is 11·22. If the length of the rope for pulling is larger than T cos θ, the boat will move fast and if the value of T sin θ is less then the boat will move faster in the forward direction. T sin θ is balanced by the helm of the boat.

EXERCISE

(A) Multiple choice questions

1. Which two quantities are vectors ?
   (a) kinetic energy, velocity
   (b) electric potential, acceleration
   (c) centripetal acceleration, temperature
   (d) electric field, force

2. Two vectors $\vec{P}$ and $\vec{Q}$ are acting parallel to each other. The two vectors will be—
   (i) coplanar vectors
   (ii) collinear vectors
   (iii) mutually reciprocal vectors
   Which of the following is correct ?
   (a) i
   (b) i and ii
   (c) i and iii
   (d) i, ii and iii

3. If the tail point and head point of a vector is same, then the vector will be—
   (i) unit vector
   (ii) null vector
   (iii) position vector
   Which of the following is correct ?
   (a) i
   (b) ii
   (c) iii
   (d) ii and iii

4. Direction of the rectangular unit vector $\hat{j}$ is—
   (i) along X-axis
   (ii) along Y-axis
   (iii) an any direction
   Which of the following is correct ?
   (a) i and ii
   (b) ii and iii
   (c) i and iii
   (d) i, ii and iii

5. If the angle between two vectors $\vec{A}$ and $\vec{B}$ is θ and $\hat{a}$ is the unit vector along the direction of $\vec{A}$ then the perpendicular projection of $\vec{B}$ on $\vec{A}$ is—
   (i) A cos θ
   (ii) B cos θ
   (iii) $\vec{B}.\hat{a}$
   Which of the following is correct ?
   (a) i
   (b) ii
   (c) i and ii
   (d) ii and iii