লাল-সবুজে দাগানো

TEXT BOOK

Physics

1st Paper

UNMESH

Medical & Dental Admission Care

3 DYNAMICS

Key Words : Reference frame, displacement, average speed, instantaneous speed or speed, average velocity, instantaneous velocity or velocity, uniform velocity, average acceleration, instantaneous acceleration or acceleration, uniform acceleration, laws of falling bodies, projectile, time of flight, range, circular motion, uniform circular motion, centripetal acceleration.




Introduction

Dynamics is a science relating motion. It deals with the motion and velocity, acceleration, force and energy of a large body, e.g. aeroplane, train, car and a small object like atom, electron etc. Dynamics establishes relation, investigates the nature of motion created due to the application of force on a body. In order to express the characteristics of motion of a moving body we need to know various terms relating motion.

Before starting to play football, cricket, tennis we should have general knowledge about how to start playing. Similarly at the beginning of learning dynamics a student should learn about velocity, acceleration, laws of falling bodies, gravitation and will be able to apply them in different graphs of motion.

After studying this chapter students will be able to—

- explain inertial frame of reference.
- explain absolute motion and relative motion.
- use differentiation and integration in describing motion.
- analyse graphs of position *versus* time and velocity *versus* time.
- explain the motion of projectile.
- explain the laws of falling bodies.
- explain uniform circular motion.

3·1 Inertial frame

Displacement, velocity, acceleration etc. of a body moving along a straight line may be explained by considering the motion along an axis of a coordinate system. In order to describe any motion a frame of reference is needed with respect to which the motion is considered. If a body is in motion along x-axis, its displacement, velocity, acceleration are respectively x, v_x, a_x. In this case it is seen that in order to know appropriate condition of motion or to determine the position of a moving body, a reference coordinate system is needed. This coordinate system is called reference frame. Simplest and well known reference frame is the Cartesian coordinate system. By it position of a particle is determined by three mutually perpendicular axes X, Y, Z.

When a butterfly enters in your study room you see it flying here and there inside the room. Suppose the butterfly sits on the book shelf of your study room. Any corner of the room may be considered as the origin and by measuring length, width and height by a tape from that origin the position of the butterfly can be ascertained. Let the position of the butterfly be determined by measuring from the corner the length of 5m, width of 3m and height of 2m. In this case the coordinate of the butterfly is (5, 3, 2). Again, if you want to determine the position of the butterfly with reference to a point outside the room, then reference system will be changed. In addition to Cartesian coordinate system the position of the body can be determined by other methods. For example, spherical or cylindrical coordinate system. That means, **for describing the motion of a body the coordinate system used in specific three dimensional space and in respect of which the motion of the body is described is called reference frame.** You can use earth surface, planet, sun, any point etc as the reference frame. But always you will have to fix them. Coordinate of a body may be fixed or may be varied with respect to time and reference frame.

If the coordinates of all the points of a body remain fixed with respect to time and reference frame, then the state of that body is called **rest.** If the coordinate of any point is changed with respect to time and reference frame then that state of the body is called **motion.**

In order to identify the position of anything on the surface of the earth or in the universe we need to fix a point. This point is called **origin** or **reference point** and by comparing the position and condition of rest and motion of a body determined with reference to a rigid body is called the **reference frame.**

If you want to express the position of a car standing on a bridge by reference frame, then you will have to consider a coordinate system. In figure 3·1, position of a car on a bridge has been shown in two dimensional reference system considering distance OP along X-axis and distance OQ along Y-axis with reference to the origin O. In this case OP = $x = 2$ unit and OQ = $y = 2$ unit i.e., the coordinate of this car, according to fig. 3·1 will be (2, 2).

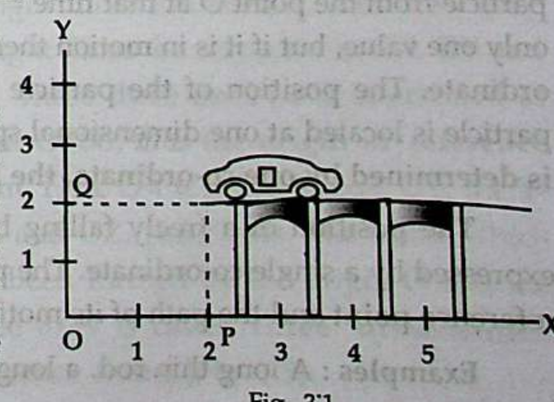


Fig. 3·1

When you sit on the dining table for taking food and see a hanging bulb above the table, then what will you do to locate the position of the bulb ? Since the bulb is not on the floor, nor on the wall, it is hanging, so you need to draw three straight lines OX, OY and OZ in order to locate the bulb. Here a three dimensional reference frame has been formed with the origin and the three axes. In fig. 3·2 the position of the bulb has been indicated. According to the figure OR = WT = x, OT = RW = y and OU = VW = z. Then the coordinate of the bulb will be (x, y, z). For determining relative rest and relative motion reference point (or standard point) and reference frame (or standard frame) is needed.

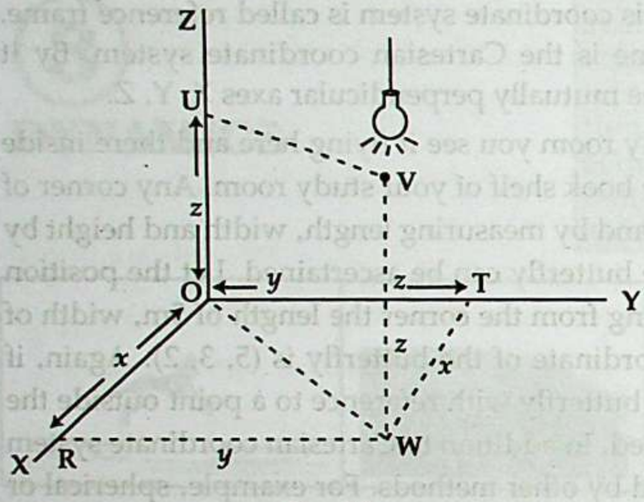


Fig. 3·2

(1) One dimensional reference frame : Suppose a particle is moving along straight line OX. The positions of the particle at different times is determined with respect to a fixed point. This fixed point with respect to which the position of the particle is determined is called **reference point or index point.** In the figure O has been taken as the reference point. The straight line OX is called X-axis. One dimensional reference frame has been formed with the reference point O and X-axis. By the help of this frame the position of the particle at any time can be determined [Fig. 3·3].

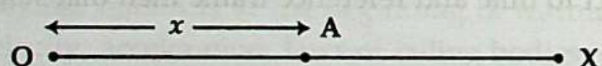


Fig. 3·3

Suppose the particle is at position A at a particular time. The distance of the particle from the point O at that time = OA = x. If the particle is at rest, then x will have only one value, but if it is in motion then x will have different values. Here x is called co-ordinate. The position of the particle is determined by only one co-ordinate, so the particle is located at one dimensional space. **If position of different particles of a body is determined by one co-ordinate, the body is called one dimensional body.**

The position of a freely falling body at different times is different and can be expressed by a single co-ordinate. The point from where the body starts falling is called **reference point** and the path of its motion will be considered X-axis.

Examples : A long thin rod, a long thin thread, a hanging thread etc. may be taken as one dimensional objects.

(2) Two dimensional reference frame : Suppose a particle is on a plane and let the particle be in motion. So, its position will be different at different times. In order to designate its position two mutually perpendicular straight lines are needed. OX and OY in the figure are two such straight lines. These two lines have met at O. So, O is the **reference point or origin.** Here OX is called X-axis and OY is called Y-axis. A frame has been formed with this origin and the two axes. **It is called two dimensional reference frame** [Fig. 3·4].

Let the particle be at position A at a particular time. From A a normal AM is drawn on OX and another normal AN is drawn on OY.

So, OM = AN = x ; AM = ON = y.

Here the position of A is designated by two co-ordinates x and y. In other words, A is a point whose co-ordinate is x and y. Hence, **if different particles of a body have two co-ordinates, the body is called two dimensional body.**

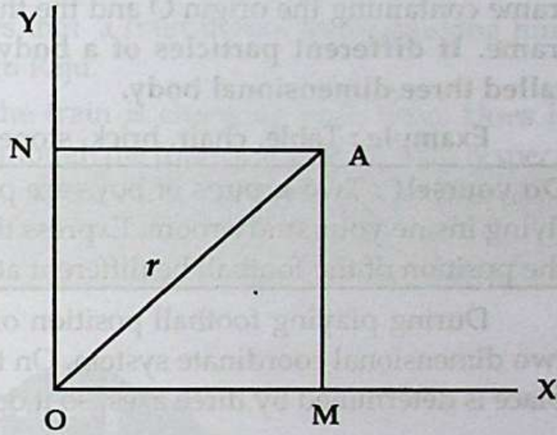


Fig. 3·4

Let us join OA. If OA = r, there will be many points C, D, E, F etc on the plane whose distance, from the point O, will be r.

Examples : A moving football in a field is in motion in two dimensional space. Thin paper, thin metal sheet etc. are two dimensional objects.

(3) Three dimensional reference frame : Let a particle be situated in a room full of air. In order to locate the position of the particle three mutually perpendicular straight lines are needed. Let the three straight lines be respectively OX, OY and OZ. These three lines have intersected at point O. So, O is **the origin or reference point.** Here OX is called X-axis, OY, the Y-axis and OZ, the Z-axis. The frame which has been formed by the origin O and the three axes is called **three dimensional reference frame** [Fig. 3·5].

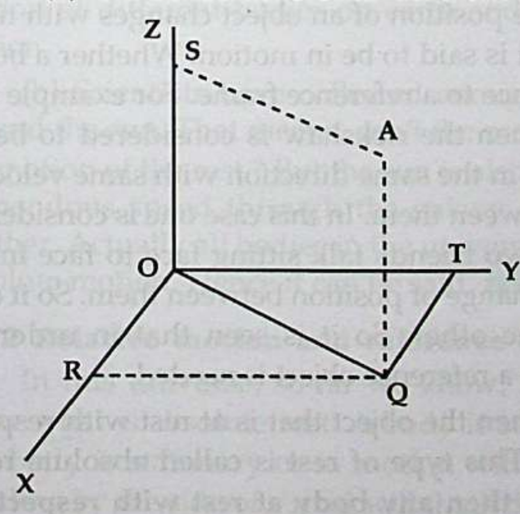


Fig. 3·5

Let the particle be at position A at a particular time. Let us draw a normal AQ from A on the plane XY. Similarly, normal QR is drawn on OX and normal QT is drawn on OY from the point Q. Hence,

$OR = QT = x$

$OT = RQ = y$

and $OS = AQ = z$

Here the co-ordinates of A has been designated by co-ordinates x, y and z. The frame containing the origin O and the three co-ordinates is called the three dimensional frame. **If different particles of a body are located in this frame, then the body is called three dimensional body.**

Example : Table, chair, brick, stone etc. are three dimensional objects.

Do yourself : Two groups of boys are playing football in the field, again a butterfly is flying inside your study room. Express these two events in reference frame system. Will the position of the football be different at different times ?

During playing football position of the football changes in a plane, so it is called two dimensional coordinate system. On the other hand position of the butterfly in every place is determined by three axes, so it occurs in three dimensional coordinate system.

3·2 Absolute motion and relative motion

3·2·1 Absolute motion

When you go to college by walking through the road, you see many moving cars, rickshaws, and many trees around you. In this case cars and rickshaws are in motion whereas trees are at rest. Again when you see vacuum cleaner cleaning the floor then distance and direction of every object around the cleaner change. That means with the passage of time position of the cleaner changes. Here we say that the vacuum cleaner is moving with respect to surroundings. When the position of an object changes with time with respect to the surroundings then the object is said to be in motion. Whether a body is in motion or at rest is understood with reference to a reference frame. For example if a rickshaw is in motion with respect to a tree then the rickshaw is considered to be in motion. Again if the rickshaw and the car move in the same direction with same velocity then there will not be any change of distance between them. In this case one is considered in motion with respect to the other. Again if two friends talk sitting face to face in an aeroplane then with time there will not be any change of position between them. So it can be said that one is at rest with respect to the other. So it is seen that in order to determine whether a body is at rest or in motion a reference object is needed.

If the reference object is actually at rest, then the object that is at rest with respect to the reference object is also actually at rest. This type of rest is called absolute rest. **When the reference object is at absolute rest then any body at rest with respect to that object is said to be at absolute rest.** Again, **when the reference object is at absolute** rest and any body is in motion then that motion is called absolute motion. Whatever we see in the universe, viz. moon, planet, satellite, earth all these are moving around the sun. So it can be said that all objects on earth are not at rest—all bodies are in motion. When we try to know whether a body is at rest or in motion then we consider it with respect to a body apparently at rest. It can be said "all rests in this universe are relative and all motions are relative. No motion is absolute, nor absolute any rest."

Observe the following examples and see how positions of objects change with time.

(a) Standing on the platform Raju observes that a train moves away crossing him [Fig. 3·6]. So the train is in motion with respect to Raju.

In this case distance between Raju and the train is changing with time. Does it mean that the train is not in absolute motion ? Although the train is in motion with respect to Raju but the earth itself rotates round the sun, so the earth can never be considered at

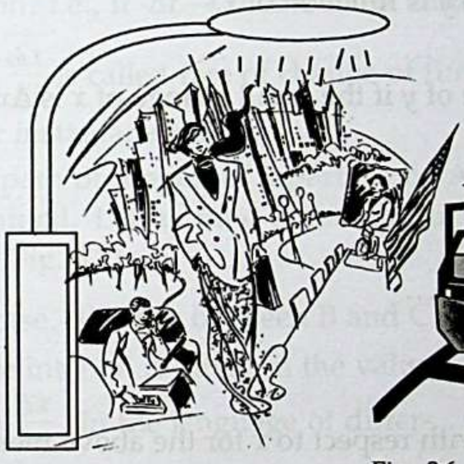


Fig. 3·6

absolute rest. In this case motion of the train is not absolute motion. Similarly, you think motion of different bodies on earth and try to understand absolute rest and absolute motion.

(b) Scientists have collected enough information about the rotation of the earth around the sun. That means, can't the motion of the train be determined by determining the motion of the rest ? But the sun is also not at rest. The whole universe is moving with tremendous speed through the galaxy. Again galaxies are in motion relative to one another. Actually all bodies in the universe are in motion. So it is impossible to determine absolute motion. Hence it can be said, rest or motion of a body is always relative.

3·2·2 Relative motion and reference frame

In this universe, so far we know, no body is stationary; that means, we do not know what is absolute rest. A body is stationary or in motion we understand whether that body is statonary or in motion with respect to a body. For example, an observer standing by the side of a road will think that a passenger in a moving train is moving. On the other hand, a pasenger in a moving train will think a tree standing by the side of road is moving in the opposite direction of the train. Again, two passengers in two different trains moving with same velocity in the same direction will think that both of them are stationary. But a passenger in a train will think that a passenger in a train moving in the opposite direction is moving faster. All these are relative. That means, all rests and motions are relative rests and relative motions respectively.

It is not possible to determine the position of a body without the help of another body. This other body is called index body or index frame.

3·3 Preliminary idea of differentiation and integration to describe motion

3·3·1 Differentiation

Suppose a quantity is dependent on another quantity, then in mathematical language this dependent quantity becomes function of the other quantity.

If quantity y is dependent on x, then y is function of x.

i.e., $y = f(x)$ ... ... (3.1)

Again, for exceedingly small change of y if the small change of x is Δx, then for the small change of y and x we can write,

$$y + \Delta y = f(\Delta x + x) \quad \ldots \quad \ldots \quad \ldots \quad (3.2)$$

$$\text{or, } \Delta y = f(\Delta x + x) - y$$

$$\text{or, } \Delta y = f(\Delta x + x) - f(x) \quad \ldots \quad \ldots \quad \ldots \quad (3.3)$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(\Delta x + x) - f(x)}{\Delta x} \quad \ldots \quad \ldots \quad \ldots \quad (3.4)$$

Here $\frac{\Delta y}{\Delta x}$ is the rate of change of y with respect to x for the above mentioned small change. When the value of Δx tends to zero, then equation (3.4) can be written as,

$$\underset{\Delta x \to 0}{\text{Lt}} \frac{\Delta y}{\Delta x} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \ldots \quad \ldots \quad (3.5)$$

$$\text{or, } \frac{dy}{dx} = \underset{\Delta x \to 0}{\text{Lt}} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \ldots \quad \ldots \quad \ldots \quad (3.6)$$

Here $\frac{dy}{dx}$ is the rate of change of y with respect to x for the exceedingly small change of x and $\frac{dy}{dx}$ is called differential co-efficient of y with respect to x and the method of determining the value of $\frac{dy}{dx}$ is called differentiation. For describing motion this differentiation is used.

Here $\frac{dy}{dx}$ is a symbol, which indicates the action of $\frac{d}{dx}$ on y. It is not the division of dy and dx and $\frac{dy}{dx}$ means the rate of variation of y with respect to x.

Suppose you are going with your friend in a motor cycle from Newmarket to Motijheel. While going you may notice that the motor cycle is not moving with same speed. Sometimes, it is moving slow, sometimes fast. Sometimes it is stopped by applying brake. So, the motion of the motor cycle is not uniform. But if you are asked to determine the speed of the motor cycle, then by dividing the total distance between Newmarket and Motijheel and time of travel you will find the speed. In this case this speed will mean average speed. That means, change of two positions with respect to time will give speed v, i.e., $v = \frac{\Delta x}{\Delta t}$.

$\frac{\Delta x}{\Delta t}$ is called the rate of change of x with respect to time for the above mentioned change or speed. Again, if the value of time is exceedingly small, then the speed for that time is the instantaneous speed. In this case this speed can be expressed by differentiation, i.e., if $\Delta t \to 0$, the fixed value of $\frac{\Delta x}{\Delta t}$ is obtained, then that limiting value $\underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta x}{\Delta t}$ is called rate of change of function $x(t)$ with respect to t or differential co-efficient or instantaneous speed.

Let the path of motion of a particle be AD and at a point B on this path speed is to be determined. Let at time t position of the particle is B and at time $t + \Delta t$ the position is C [Fig. 3·7].

In this case, distance between B and C is Δx and time interval is Δt. Then the value of speed, $v = \frac{\Delta x}{\Delta t}$. In the language of differentiation, this variation of x with respect to t is the magnitude of speed.

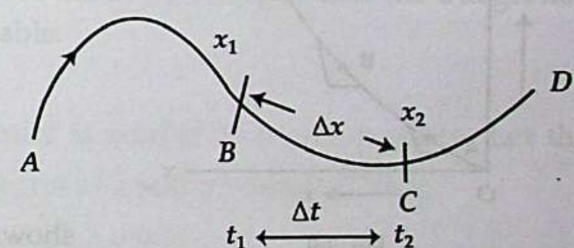


Fig. 3·7

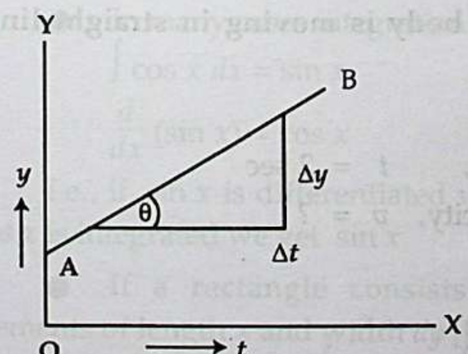


Fig. 3·8

Again, for the change Δt of t, the small Δy can be expressed graphically [Fig. 3·8]. In this case if Δy that is Δt is exceedingly small, then $\frac{\Delta y}{\Delta t}$ will indicate the slope of the straight line.

$$\therefore \quad \underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta y}{\Delta t} = \text{slope}$$

$$\text{or,} \quad \frac{dy}{dt} = \text{slope}$$

That means, by determining the slope from displacement (y) versus time (t) graph velocity can be determined from the change of displacement with change of time.

Again, let $x = f(t) = t^2$

$$\therefore \frac{dx}{dt} = \underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta x}{\Delta t} = \underset{\Delta t \to 0}{\text{Lt}} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$= \underset{\Delta t \to 0}{\text{Lt}} \frac{(t+\Delta t)^2 - t^2}{\Delta t}$$

$$= \underset{p \to 0}{\text{Lt}} \frac{(t+p)^2 - t^2}{p}, \quad \Delta t = p \text{ (suppose)}$$

$$= \underset{p \to 0}{\text{Lt}} \frac{2tp + p^2}{p}$$

$$= \underset{p \to 0}{\text{Lt}} 2t + p = 2t$$

$$\therefore \frac{d}{dt}(t^2) = 2t \quad \ldots \ldots \ldots \quad (3.7)$$

In the graph below, slopes at different points in the curve is not equal. For determining slope at a point in this type of graph an exceedingly small time interval Δt around Δx is to be considered. This slope can be shown to be equal to differential co-efficient at that point.

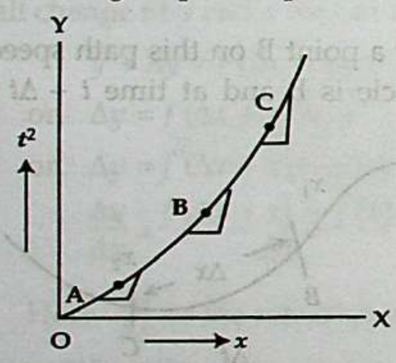


Fig. 3·9

In this case, the value of slope $= \frac{\Delta t^2}{\Delta x}$

or, $\frac{d}{dx}(t^2) = 2t$.

In fig. 3·9, a graph of t^2 versus x has been shown.

Mathematical example

1. According to the equation $s = \frac{1}{3}t^3 + 3t$, a body is moving in straight line. Calculate the velocity after 2 sec.

Here,
Time, $t = 2$ sec
Velocity, $v = ?$

Let the velocity $= v$

We know, $v = \frac{ds}{dt}$

Now, $s = \frac{1}{3}t^3 + 3t$

Differentiating s with respect to t we get,

$$\frac{ds}{dt} = \frac{d}{dt}\left(\frac{1}{3}t^3 + 3t\right)$$

or, $v = \frac{1}{3} \times 3t^2 + 3 = t^2 + 3$

$\therefore$ After 2 sec velocity, $v = (2)^2 + 3 = 7$ unit.

3·3·2 Integration

Suppose the length of a rod is x and the rod is divided into innumerable equal segments. Length of one of these segments = dx. If we add all these small segments we can find the length x of the entire rod.

$$\therefore \quad \Sigma dx = x \quad \ldots \quad \ldots \quad \ldots \quad (3.8)$$

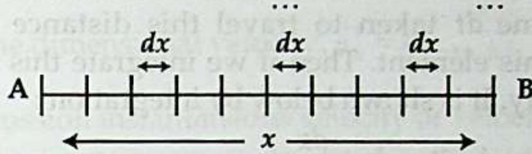


Fig. 3·10

Here symbol Σ expresses integration.

Equation (3·8) can be written in the following way,

$$\int dx = x \quad \ldots \quad \ldots \quad \ldots \quad (3.9)$$

'$\int$' symbol also indicates integration. It is seen from this equation that integral value of dx is equal to x. So, we can say that integration is a form of summation. In order to bring equality in both sides of the above equation a constant of integration c is added and written as $\int dx = x + c$.

Again, let integral value of a function $f(t)$ be $\int f(t)dt = A(x)$

Then $f(t)$ is called the integral quantity. dt after $f(t)$ indicates that the integration is to be done with respect to t. It is called variable.

1. Application of integration in some cases

- General integration of vector quantity is similar to scalar quantity. Let the vector $\vec{A}(t) = \hat{i}\, A_x(t) + \hat{j}\, A_y(t) + \hat{k}\, A_z(t)$ be integral of a scalar variable t, then

$$\int \vec{A}\,(t)\,dt = \hat{i} \int A_x(t)\,dt + \hat{j} \int A_y(t)dt + \hat{k} \int A_z(t)\,dt.$$

  It is called the indefinite integral of $\vec{A}(t)$.

- In many cases integration is opposite to differentiation; for example—

$$\int \cos x\, dx = \sin x$$

$$\frac{d}{dx}(\sin x) = \cos x$$

  i.e., if $\sin x$ is differentiated we get $\cos x$ and if $\cos x$ is integrated we get $\sin x$.

- If a rectangle consists of innumerable elements of length x and width dy [Fig. 3·11] then area of the rectangle will be, $\int x dy = x \int dy = xy$

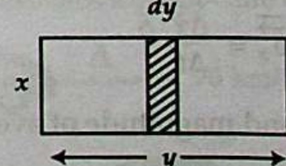


Fig. 3·11

- If limit of width is mentioned, then area $= \int_0^y x\, dy = x\,[y]_0^y = x\,(y - 0) = xy$

2. Application of integration in case of velocity

While calculating velocity of a moving body we divide displacement by total time and get velocity. This velocity is nothing but average velocity. Because, while moving at one time motion of the body is uniform and in other time it is non-uniform. So, if the moving path is divided into innumerable width or elements and each of these elements dx is divided by the time dt taken to travel this distance (element), then we can determine velocity for this element. Then if we integrate this velocity dv for the entire path we get actual velocity. It is shown below by integration :

For the element dx, velocity, $dv = \frac{dx}{dt}$

Again for the entire path, velocity $= \int dv = v + c$ ... ... (3.10)

If the magnitude of the velocity is definite, then by solving the above integration within that limit we get actual velocity. For example, if the magnitude of velocity is $v = v$ from $v = 0$, then by integrating equation (3.10) total velocity is obtained.

In this case, total velocity $= \int_0^v dv = [v]_0^v = (v - 0) = v$

So, it can be said that summation of very small elements is integration. By a real example we can get the idea. For example, a person when reads news in front of the camera of the TV studio, then his picture is divided into innumerable elements, again when it reaches on the screen then those elements of the picture get united and transforms into full picture. In this case, formation of very small elements of the picture is differentiation and the process of unification of these small elements on the TV screen is integration.

3.4 Different quantities related to motion

(1) Average velocity : The quantity which is obtained by dividing the total displacement by time taken for that displacement is called average velocity.

Explanation : Let total displacement of a body be $\Delta \vec{r}$ in time interval of Δt.

$\therefore$ Average velocity, $\vec{\bar{v}} = \frac{\Delta \vec{r}}{\Delta t}$

If the motion is one dimensional and if the body is in motion along X-axis, then there exists only one component of velocity. The component will be—

$\vec{\bar{v}}_x = \frac{\Delta x}{\Delta t} \hat{i}$ [$\because$ in one-dimensional reference, $\vec{r} = x\hat{i}$]

and magnitude of average velocity, $\bar{v}_x = \frac{\Delta x}{\Delta t}$.

(2) Instantaneous velocity or velocity : If the time interval approaches zero, the rate of change of displacement with time is called instantaneous velocity or simply velocity. Instantaneous velocity means the velocity of a body at a particular time. In order to find the instantaneous velocity of a body, at an instant, it is essential to know the velocity just before and just after that instant. This time interval must be exceedingly small (almost zero).

That means, when time interval tends to zero, rate of change of displacement for that time interval is called instantaneous velocity.

So, instantaneous one dimensional velocity, $\vec{v}_x = \underset{\Delta t \to 0}{\text{Lt}} \frac{\Delta \vec{x}}{\Delta t} = \frac{d\vec{x}}{dt}$

From the above discussion instantaneous velocity or velocity may be defined as :

If the time interval tends to zero, limiting value of the velocity is called instantaneous velocity or velocity.

(3) Mean velocity : If the initial and final directions of a moving body remain same then half of their summation is called mean velocity.

Suppose, initial velocity of a body in a particular direction is $\vec{v_0}$ and final velocity is $\vec{v}$.

$\therefore$ **Mean velocity** $= \frac{\vec{v_0} + \vec{v}}{2}$

(4) Relative velocity : Generally, by the velocity of a body we understand its velocity with respect to earth. Again we can also find the velocity of a body A with respect to another body B which is in motion on earth. It is called the relative velocity of the body **A with respect to the body B.** In this case, relative velocity of a body with reference to another body is equal to the vector difference of the two velocities i.e., relative velocity of a body with reference to another body in motion on the earth's surface is equal to the vector difference of the two velocities.

Suppose the velocities of two bodies A and B are respectively v_A and v_B, then relative velocity of A with respect to B is

$$v_{AB} = v_A - v_B \quad \ldots \quad \ldots \quad \ldots \quad (3.11)$$

Again, relative velocity of B with respect to A is,

$$v_{BA} = v_B - v_A = -(v_A - v_B)$$

$$\therefore \quad v_{BA} = -v_{AB} \quad \ldots \quad \ldots \quad \ldots \quad (3.12)$$

Example (a) : Look at the figures below. In the first figure two cars are moving in the same direction [Fig. 3·12(a)] and in the second figure they are moving opposite to each

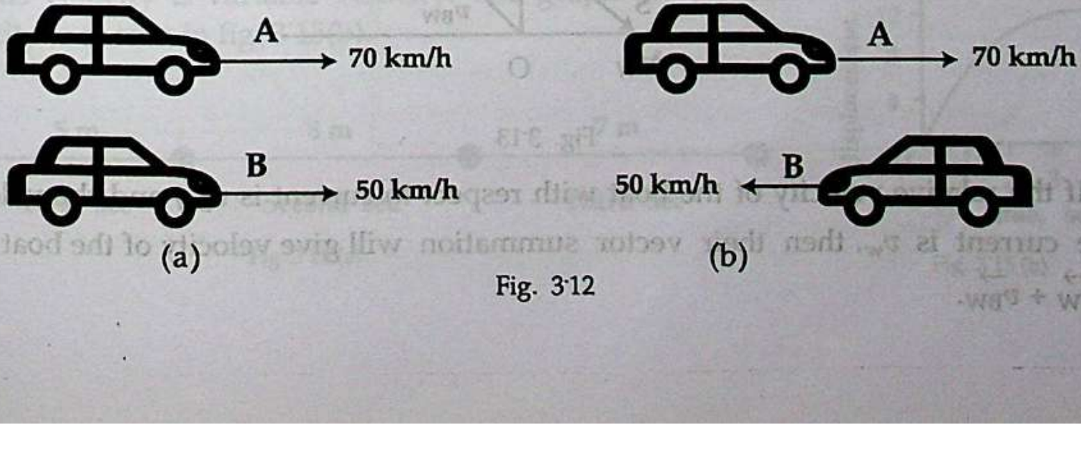


Fig. 3·12

other [Fig. 3·12(b)]. How will you determine the relative velocity between them ? Here, in case of two parallel linear motion the following two motions are to be considered for determining relative velocity.

● **Motion in same direction :** When two bodies move in the same direction along a straight line, then the relative velocity of A with respect to B will be equal to the difference of the two velocities.

That is, $v_{AB} = v_A - v_B$, if $v_A > v_B$, then the direction of v_{AB} will be along the direction of v_A. So looking from B the body A will appear moving in the forward direction with velocity v_{AB}. Again, if it is seen from A the body B will appear to be moving backward with velocity v_{BA} $(= - v_{AB})$ as direction of v_{BA} is opposite to v_{AB}.

● **Motion in opposite direction :** If two bodies start moving opposite to each other and if the velocity v_A of the body A is considered positive, then velocity v_B of the body B is to be taken as negative. Here the relative velocity of A with respect to B will be, $v_{AB} = v_A - (-v_B) = v_A + v_B$, if $v_A > v_B$.

So, looking from the body B it will appear that the body A is approaching towards or moving away from B with velocity v_{AB}. Since $v_{BA} = - v_B - v_A = -(v_B + v_A)$, so looking from the body A it will appear that the body B is approaching with velocity v_{BA} $(= - v_{AB})$ towards A or moving away from it.

Example (b) : Suppose a person from one side of a river at point O wants to go by boat exactly to the opposite side of the river at point P, then the person is to row the boat in the direction ON [Fig. 3·13].

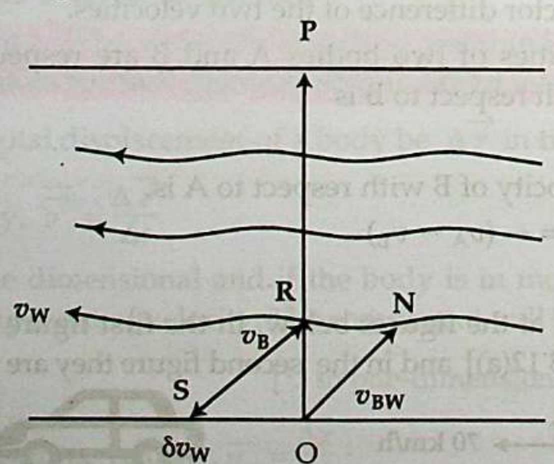


Fig. 3·13

If the relative velocity of the boat with respect to current is v_{BW} and the velocity of the current is v_w, then their vector summation will give velocity of the boat, i.e., $\vec{v}_B = \vec{v}_W + \vec{v}_{BW}$.

Conclusions : Following conclusions may be drawn from the above examples :

(1) If two bodies move in the same direction, then by subtracting their velocities relative velocity can be obtained.

(2) If two bodies move in the opposite direction, then the relative velocity is obtained by adding the two velocities.

(5) Uniform velocity : If the velocity always remains constant then the velocity is called uniform velocity.

If magnitude and direction of the force applied on a body is constant, then velocity also remains constant. This velocity of the body is the uniform velocity.

Examples : Velocity of sound, velocity of light etc.

Explanation : In fig. 3·14(a), position of a moving body along a straight line has been shown by five successive points with 1 second interval. Here the distance between two successive points is 0·25 m. According to motion the body is travelling 0·25 m in every second in the same direction and travelling same path in the same interval of time. So, the velocity of the body is uniform velocity and the magnitude of that uniform velocity is 0·25 ms^{-1}. In fig. 3·14(b), uniform velocity has been shown by a graph of displacement versus time.

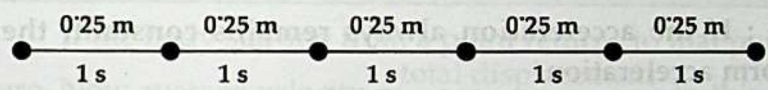


Fig. 3·14 (a)

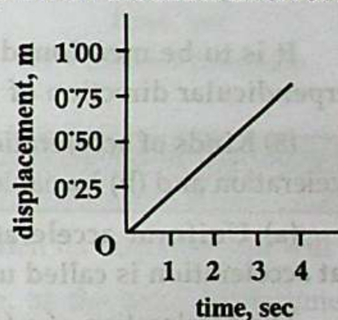


Fig. 3·14 (b)

A body has uniform velocity of 5 ms^{-1}. It means that in a particular direction the body travels 5 m in every second.

(6) Variable velocity : If the velocity of a body is different at different times then that velocity is called variable velocity. So if the magnitude or direction or both of rate of change of displacement with time changes then that rate of change of displacement is called variable velocity.

Explanation : Let a body travel 5 m in the first second, 8 m in the second second and 7 m in the third second in a particular direction [Fig. 3·15(a)]. Here the body is not travelling the same distance in the same interval of time. So this velocity is variable velocity. The graph of variable velocity is shown in fig. 3·15(b).

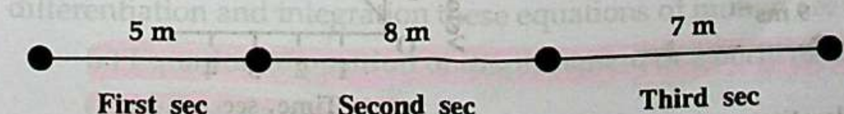


Fig. 3·15 (a)

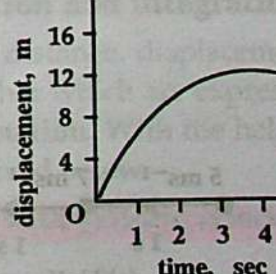


Fig. 3·15 (b)

(7) Instantaneous acceleration or acceleration : If the time interval approaches zero, the rate of change of velocity with time of a body is called the instantaneous acceleration or simply acceleration.

Explanation : Let the change of velocity at exceedingly small time interval Δt be $\Delta \vec{v}$. Then instantaneous acceleration is obtained by dividing $\Delta \vec{v}$ by the exceedingly small time during which change of velocity has taken place.

So, instantaneous acceleration or acceleration, $\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$

Hence, instantaneous accelerations may be defined as follows :

If the time interval tends to zero, the limiting value of average acceleration is equal to the acceleration.

The magnitude of acceleration is, $a = \left| \frac{d\vec{v}}{dt} \right|$

It is to be mentioned that instantaneous acceleration at any point is along the perpendicular direction of the velocity of the body at that point.

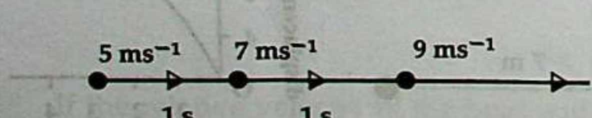
(8) Kinds of acceleration : Acceleration is divided into two kinds, viz.—(a) Uniform acceleration and (b) Variable acceleration.

(a) Uniform acceleration : If the acceleration always remains constant, then that acceleration is called uniform acceleration.

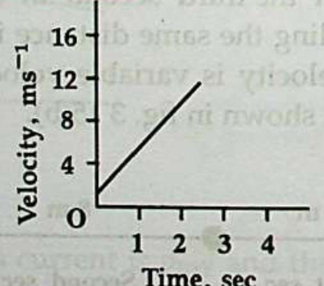
The acceleration of a freely falling body due to the action of gravitation is uniform acceleration. Equal force acts on a body of uniform acceleration. Both the magnitude and direction of uniform acceleration remain constant.

In fig. 3·16(a), uniform acceleration is shown by successive change of velocity with time along a straight line. In fig. 3·16(b) uniform acceleration has been shown by a graph. Here the value of uniform acceleration is 2 ms^{-2}. In case of uniform acceleration the graph is a straight line and the slope becomes constant.

The uniform acceleration of a body is 10 ms^{-2} means that the velocity of the body changes in every second by 10 ms^{-1} in the same direction.



(a) Uniform acceleration



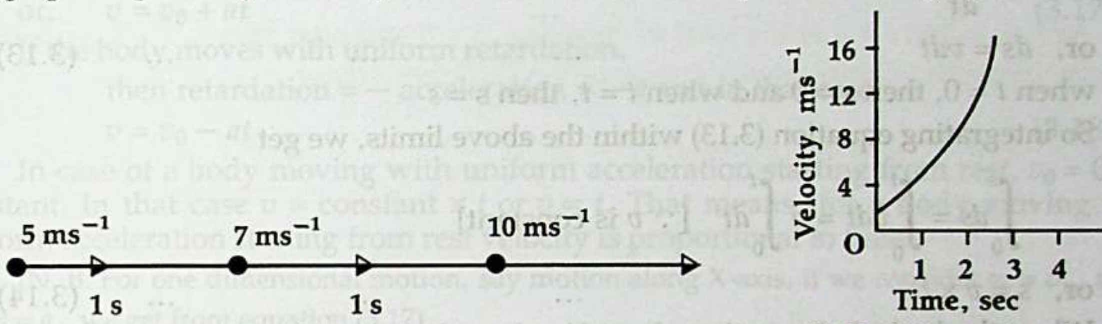
(b) Uniform acceleration

Fig. 3·16

(b) Variable acceleration : When the acceleration of a body changes with time, the acceleration is called variable acceleration.

Variable acceleration can be produced due to the change of magnitude and direction or magnitude or direction of acceleration. The acceleration of bus, train, car etc are examples of variable acceleration.

In fig. 3'17(a) and 3'17(b) variable accelerations have been shown by straight line and graph respectively. The slope is different at different points of the graph.



(a) Variable acceleration (b) Variable acceleration

Fig. 3'17

Work : Average velocity of a body is zero but can the average speed of that body be zero.? explain.

If the body starting from a point returns to that point then its displacement will be zero. Now average velocity $= \frac{\text{total displacement}}{\text{total time}}$. In this case, as the total displacement is zero, hence average velocity will be zero.

Again, we know, average speed $= \frac{\text{total distance travelled}}{\text{total time}}$.

In this case traversed distance is not zero, hence average speed will not be zero. So, although avarage velocity is zero, but average speed is not zero.

Do yourself : Velocity of a body is zero but acceleration is not zero—is it possible ? explain.

3'5 Derivation of equations of motion using differentiation and integration

In the previous section, we have discussed quantities like distance, displacement, velocity, acceleration etc. These quantities are related to each other which are expressed by some equations. **These equations are called equations of motion.** With the help of differentiation and integration these equations of motion are derived below.

(a) Equation of position or displacement of a body moving with uniform velocity ($s = vt$) or $x = x_0 + v_x t$.

Let an object move in a particular direction with uniform velocity.

Suppose, the uniform velocity of the body = v
the initial displacement of the body = 0
distance travelled at time t sec = s

If distance travelled at exceedingly small time dt sec is ds then distance travelled at $t + dt$ sec = $s + ds$

Now, dt being exceedingly small, from the definition of velocity, we get

$$v = \frac{ds}{dt}$$

or, $ds = vdt$ ... ... (3.13)

when $t = 0$, then $s = 0$ and when $t = t$, then $s = s$

So integrating equation (3.13) within the above limits, we get

$$\int_0^s ds = \int_0^t vdt = v\int_0^t dt \quad [\because v \text{ is constant}]$$

or, $s = v \times t$ ... ... (3.14)

When the body is in motion along X-axis and at the start of motion when $t = 0$, then $x = x_0$ and when $t = t$, then $s = x$ and velocity becomes $v = v_x$ [Fig. 3·18], then integrating the equation (3·13) within the above limit we get,

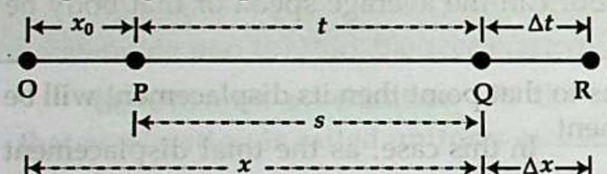


Fig. 3·18

$$\int_{x_0}^{x} ds = v_x \int_0^t dt$$

or, $[s]_{x_0}^{x} = v_x\,[t]_0^t$

or, $x - x_0 = v_y t$

or, $x = x_0 + v_x t$ ... ... (3.15)

(b) Equation of motion relating final velocity, acceleration and time ($v = v_0 + at$ or $v_x = v_{x0} + a_x t$)

Suppose a body is moving in a direction with initial velocity v_0 and with uniform acceleration $\vec{a}$.

Let the velocity in exceedingly small time dt be increased from v to $v + dv$ [Fig. 3·19]. Then according to the definition of acceleration, we get

$$a = \frac{dv}{dt}$$

or, $dv = adt$ ... ... (3.16)

Now, when $t = 0$, then $v = v_0$ and $t = t$, then $v = v$

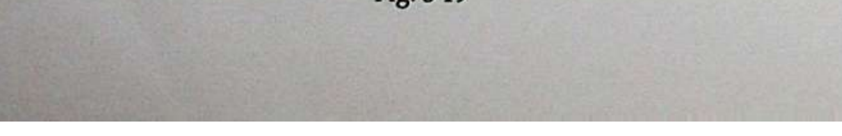


Fig. 3·19

So, integrating equation (3.18) within this limit, we get,

$$\int_{v_0}^{v} dv = \int_{0}^{t} a dt$$

or, $[v]_{v_0}^{v} = a[t]_0^t$ $[\because a \text{ is constant}]$

or, $v - v_0 = at$

or, $v = v_0 + at$ ... ... (3.17)

If the body moves with uniform retardation,

then retardation = − acceleration = − a and in that case

$v = v_0 - at$ ... ... (3.18)

In case of a body moving with uniform acceleration starting from rest, $v_0 = 0$, a = constant. In that case v = constant × t or $v \propto t$. That means, for a body moving with uniform acceleration starting from rest velocity is proportional to time.

[N. B. For one dimensional motion, say motion along X-axis, if we consider $v_0 = v_{x0}$, $v = v_x$ and $a = a_x$, we get from equation (3.17),

$v_x = v_{x0} + a_x t$ ... ... (3.19)

Similarly the equation (3.18) will be changed. In case of motion along Y or Z instead of x respectively y or z is to be used.]

(c) Equation of motion relating position or displacement, acceleration and time $s = v_0 t + \frac{1}{2} at^2$ or, $x = x_0 + v_{x_0} t + \frac{1}{2} a_x t^2$

Suppose, a particle at initial velocity v_0 is moving in a particular direction with uniform acceleration a.

The particle aquires velocity v after time t traversing a distance s and in the same direction after moving a distance ds in excedingly small time dt acquires velocity $v + dv$ [Fig. 3·20]

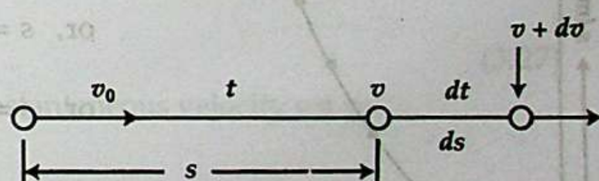


Fig. 3·20

Now, according to the definition of instantaneous acceleration we get,

$a = \frac{dv}{dt}$

or, $dv = adt$ ... ... (3.20)

When $t = 0$, then $v = v_0$ and when $t = t$, then velocity $v = v$. Within this limit integrating both sides of the equation (3.20), we get,

$$\int_{v_0}^{v} dv = \int_{0}^{t} adt$$

or, $[v]_{v_0}^{v} = a[t]_0^t$

or, $v - v_0 = a(t - 0)$

or, $v = v_0 + at$ ... ... (3.21)

Again, according to the definition of instantaneous velocity,

$$v = \frac{ds}{dt}$$

or, $ds = vdt$

or, $ds = (v_0 + at)dt$ [using equation 3.21]

or, $ds = v_0dt + atdt$ ... ... (3.22)

Again, when $t = 0$ at the start of counting, $s = 0$ and after time t, $s = s$; within this limit, by integrating both sides of the equation (3.22) we get,

$$\int_0^s ds = \int_v^t v_0 dt + \int_v^t atdt$$

or, $\int_0^s ds = v_0 \int_v^t dt + a \int_v^t tdt$

or, $[s]_0^s = v_0\,[t]_0^t + a\,[\frac{t^2}{2}]_0^t$

or, $(s - 0) = v_0\,(t - 0) + a\left(\frac{t^2}{2} - \frac{0^2}{2}\right)$

or, $s = v_0t + \frac{1}{2}at^2$ ... ... (3.23)

Again, if the body is in motion from rest with uniform acceleration, then we get,

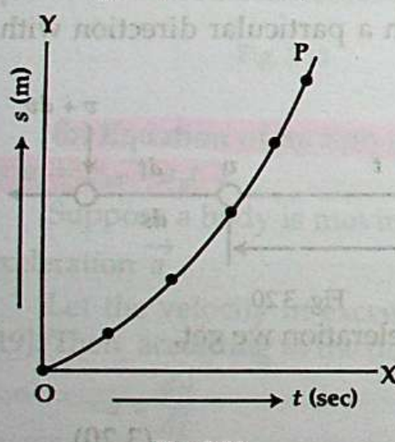


Fig. 3'21

$s = 0 \times t + \frac{1}{2}at^2$

or, $s = 0 + \frac{1}{2}at^2$

or, $s = \frac{1}{2}at^2$

or, $s = \text{constant} \times t^2$ $\left[\because \frac{1}{2}a = \text{constant}\right]$

or, $s \propto t^2$

That means, starting from rest and moving with uniform acceleration distance transversed by a body is directly proportional to the square of time.

In fig. 3'21 a graph of displacement versus time has been shown.

If the body moves along X-axis and at time $t = 0$ the initial velocity $= v_{x0}$, and at any other time t final velocity $= v$ and a_x is uniform acceleration, then equation (3.23) can be written as, $s = v_{x0}t + \frac{1}{2}a_xt^2$.

Now at time $t = 0$ if the initial position of the body is x_0 and at time t its position is x, then $s = x - x_0$.

In that case,

$$s = x - x_0 = v_{x0}t + \frac{1}{2}a_x t^2$$

$$\text{or, } x = x_0 + v_{x0}t + \frac{1}{2}a_x t^2 \quad \ldots \quad \ldots \quad \ldots \quad (3.24)$$

If the body, instead of moving with uniform acceleration a, moves with uniform retardation a, then

retardation $= -$acceleration $= -a$, from equation (3.24) we get,

$$s = v_0 t - \frac{1}{2}at^2 \quad \ldots \quad \ldots \quad \ldots \quad (3.25)$$

Suppose a body is moving along X-axis with uniform acceleration a_x and at the start of counting i.e., at $t = 0$ initial position of the body $= x_0$, velocity $= v_{x_0}$ and after time t position $= x$, velocity $= v_x$. Now, if the body travels a distance dx in exceedingly small time dt and acquires velocity $v_x + dv_x$ then according to the definition of instantaneous acceleration we get,

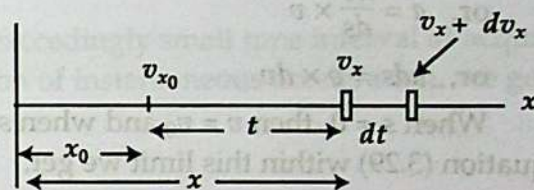


Fig. 3·22

$$a_x = \frac{dv_x}{dt}$$

$$\text{or, } dv_x = a_x dt \quad \ldots \quad \ldots \quad (3.26)$$

Now integrating the equation (3.26) in appropriate limit i.e., $t = 0$ and $t = t$ and corresponding velocities are v_{x_0} and v_x, we get

$$\int_{v_{x_0}}^{v_x} dv_x = \int_0^t a_x dt$$

$$\text{or, } [v_x]_{v_{x_0}}^{v_x} = a_x [t]_0^t$$

$$\text{or, } v_x - v_{x_0} = a_x (t - 0)$$

$$\text{or, } v_x = v_{x_0} + a_x t \quad \ldots \quad \ldots \quad \ldots \quad (3.27)$$

Again, according to the definition of instantaneous velocity we get,

$$v_x = \frac{dx}{dt}$$

$$\text{or, } dx = v_x dt$$

$$\text{or, } dx = (v_{x0} + a_x t)\, dt \quad \ldots \quad \ldots \quad (3.28) \quad \text{[using equation. 3.27]}$$

Integrating both sides of the equation (3.28) within limits x and x_0 and x and $t = 0$ and t we get,

$$\int_{x_0}^{x} dx = \int_0^t (v_{x_0} + a_x t)\, dt$$

$$\text{or, } [x]_{x_0}^{x} = v_{x_0} [t]_0^t + a_x \left[\frac{t^2}{2}\right]_0^t$$

$$\text{or, } x - x_0 = v_{x_0} t + a_x \times \frac{t^2}{2}$$

$$\text{or, } x = x_0 + v_{x_0} t + \frac{1}{2}a_x t^2$$

(iv) Relation between initial velocity of a body moving with uniform acceleration, final velocity and distance ($v^2 = v_0^2 + 2as$ or, $v_x^2 = v_{x0}^2 + 2a(x - x_0)$)

Let initial velocity of a body moving along a straight line with uniform acceleration a be v_0; after time t, final velocity is v and during that time the body travels a distance s. An equation relating v, v_0, a and s is to be established.

According to the definition of instantaneous acceleration,

$$a = \frac{dv}{dt}$$

or, $a = \frac{dv}{ds} \times \frac{ds}{dt}$

or, $a = \frac{dv}{ds} \times v$

or, $ads = v \times dv$ ... ... (3.29)

When $s = 0$, then $v = v_0$ and when $s = s$, then $v = v$; by integrating both sides of the equation (3.29) within this limit we get,

$$\int_0^s ads = \int_{v_0}^{v} vdv$$

or, $a\int_0^s ds = \int_{v_0}^{v} vdv$

or, $a[s]_0^s = \left[\frac{v^2}{2}\right]_{v_0}^{v}$

or, $a(s - 0) = \left(\frac{v^2}{2} - \frac{v_0^2}{2}\right)$

or, $as = \frac{v^2 - v_0^2}{2}$

or, $2as = v^2 - v_0^2$

or, $v^2 = v_0^2 + 2as$ ... ... (3.30)

Again, if the body starts moving from rest, we get

$$v^2 = 0^2 + 2as$$

or, $v^2 = 2as$

or, $v^2 = \text{constant} \times s$ [$\because$ $2a$ is constant]

or, $v^2 \propto s$

or, $v \propto \sqrt{s}$

That means, starting from rest final velocity of a body moving with uniform acceleration is proportional to the square root of distance.

Derivation of $v_x^2 = v_{x0}^2 + 2a_x (x - x_0)$

Suppose a body is moving along X-axis with uniform acceleration a_x.

At start, i.e., when time, $t = 0$, then the initial position = x_0 and after time t velocity = v_x and position = x

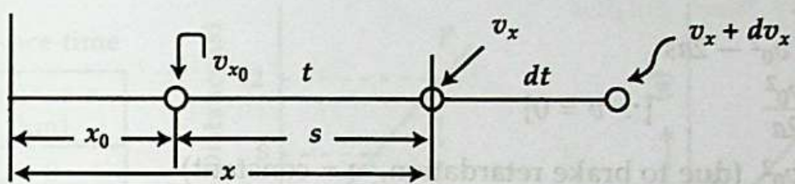


Fig. 3·23

Now, along that X-axis if the body after exccedingly small time interval dt acquires velocity $v_x + dv_x$, then according to the definition of instantaneous acceleration, we get

$$a_x = \frac{dv_x}{dt}$$

or, $$a_x = \frac{dv_x}{dx} \times \frac{dx}{dt}$$

or, $$a_x = \frac{dv_x}{dx} \times v_x$$

or, $$a_x dx = v_x \times dv_x \quad \dots \quad \dots \quad \dots \quad (3.31)$$

When $x = x_0$, then $v = v_{x0}$ and when $x = x$, then $v = v_x$; within this limit by integrating both sides of the equation (3.31), we get

$$\int_{x0}^{x} a_x dx = \int_{v_{x0}}^{v_x} v_x dv_x$$

or, $$a_x \int_{x0}^{x} dx = \int_{v_{x0}}^{v_x} v_x dv_x$$

or, $$a_x [x]_{x0}^{x} = \left[\frac{v_x^2}{2}\right]_{v_{x0}}^{v_x}$$

or, $$a_x (x - x_0) = \left(\frac{v_x^2}{2} - \frac{v_{x0}^2}{2}\right)$$

or, $$a_x (x - x_0) = \frac{v_x^2 - v_{x0}^2}{2}$$

or, $$v_x^2 - v_{x0}^2 = 2a_x (x - x_0)$$

or, $$v_x^2 = v_{x0}^2 + 2a_x (x - x_0)$$

Inquisitive work : If the velocity of a car is doubled then how many times the distance should be to stop the car by applying brake ? explain.

Let retardation a be generated by applying brake of a car moving with velocity v_0; consequently the car stopped after travelling distance s. Here final velocity $v = 0$.

We know,

$$v^2 = v_0^2 - 2as$$

or, $$s = \frac{v_0^2}{2a} \qquad [\because v = 0]$$

or, $s \propto v_0^2$ (due to brake retardation, a = constant)

Hence, if the initial velocity of the car is doubled, then the distance for stopping the car, $s = (2)^2 = 4$ times.

Perceptional work : Why a body moving with uniform speed does not have acceleration?

3·6 Position-time and velocity-time graphs

Change of position of a body can be expressed by position-time graph and change of velocity can be expressed by velocity-time graph.

3·6·1 Position-time graphs

With the passage of time position of a moving body changes. This relation can be expressed by a graph. In this case, time (t) is placed along X-axis and change of position (Δx) along Y-axis. This graph is called **position-time graph**. From this graph velocity is determined. In the following sections, uniform motion, non-uniform motion and motion along straight lines have been presented for a moving body.

1. If a body is located to a particular distance from a fixed point reference, then the distance between point and the body is called position.

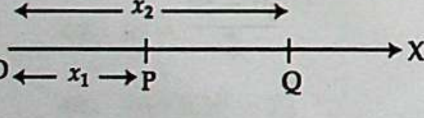


Fig. 3·24

Suppose, a body is moving along X-axis. When the body is at point P then distance of the body from the origin O is x_1 and when the body remains at point Q, then the distance is x_2. Motion of such a body along X-axis can be represented by figure 3·24.

Again, suppose a body is located at point A at time t_1 and at point B at time t_2. Coordinates of the points A and B along X-axis are respectively x_1 and x_2. In this case time interval t_2 and t_1 is $\Delta t = t_2 - t_1$ [shown in figure 3·25].

2. Through a graph we can discuss the motion of a body.

In this process a graph is drawn by placing displacement, velocity or acceleration along Y-axis and time along X-axis.

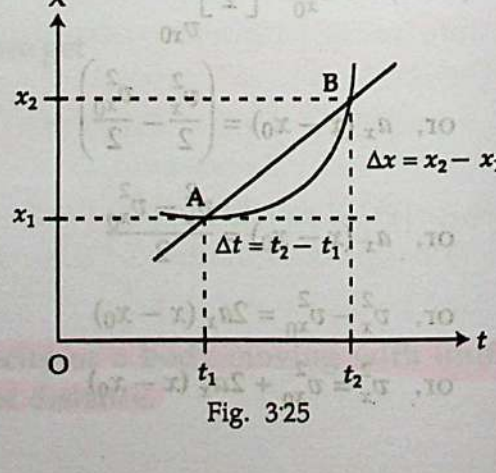


Fig. 3·25

3.6.2 Distance-time graphs

(i) Distance-time graph (In case of uniform velocity) :

Motion of a motor cycle in every two minutes along a plane road is shown in table 3.1.

Table 3.1 : Distance-time

Time t (min)	Distance s (km)
0	0
2	4
4	8
6	12

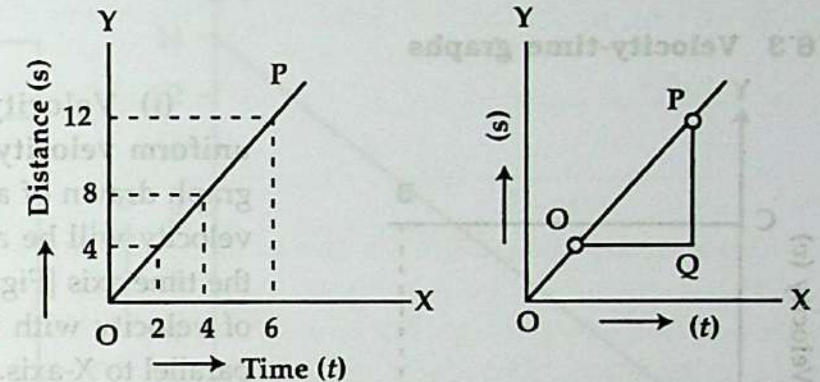


Fig. 3.26

A body moving with uniform velocity covers equal distance in equal interval of time. The graph of distance versus time will be a straight line OP passing through the origin O [Fig. 3.26].

$$\text{Slope of the straight line OP} = \frac{PQ}{OQ} = \frac{\text{Distance}}{\text{Time}} = \text{Velocity}$$

So, slope of uniform velocity of the body will be equal to the slope of the distance-time graph. That means, magnitude of the velocity at any time is equal to the slope at that point.

Work : Draw a graph in a graph paper taking data of table 3.1. From the above graph, calculate the distance travelled in 10 minutes and also the velocity.

(ii) Distance-time graph (In case of non-uniform velocity) :

A body moving with non-uniform velocity does not cover equal distance in equal interval of time. Graph of position (x) and time (t) will be a curve. A velocity at any time at any position can be obtained from the slope drawn at that position. In figure 3.27, velocity has been determined at point P.

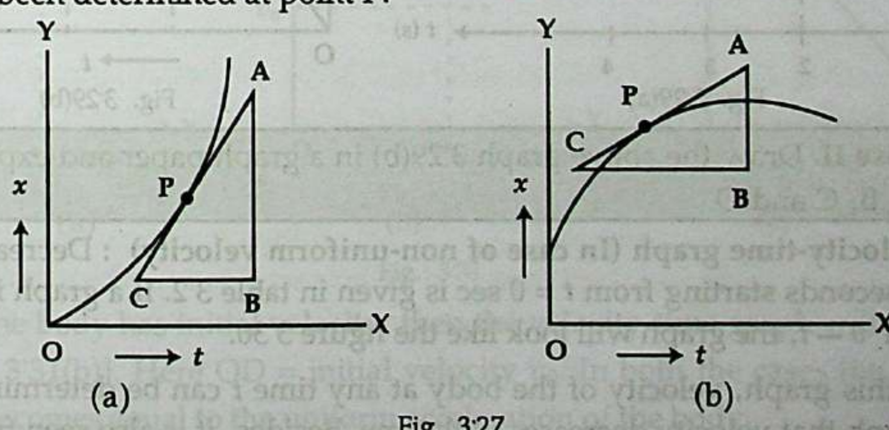


Fig. 3.27

Slopes at different points, in this graph, are different i.e., velocity is different at different times. The magnitude of the slope at a point gives non-uniform velocity at that time.

According to figure, velocity at $P = \frac{AB}{CB} = \frac{x}{t}$.

3·6·3 Velocity-time graphs

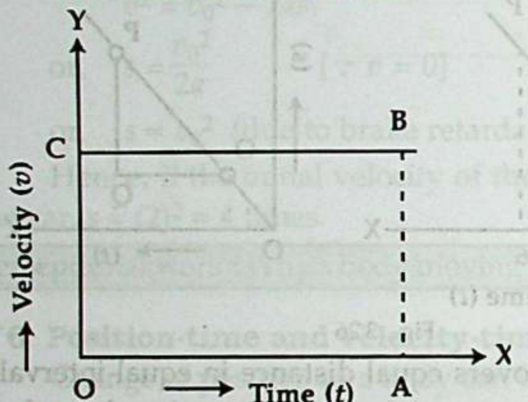


Fig. 3·28

(i) Velocity-time graph (In case of uniform velocity) : The velocity versus time graph drawn of a body moving with uniform velocity will be a straight line CB parallel to the time axis [Fig. 3·28]. As there is no change of velocity with time, so the graph becomes parallel to X-axis.

Area of the rectangle OABC subtended by the velocity and time axes = OC × OA = $vt = s$

That means, distance travelled by the body is equal to the area covered by velocity and time axes of the velocity-time graph.

Work exercise I. Draw the following graph [3·29(a)] in a graph paper and explain velocities at point A and B.

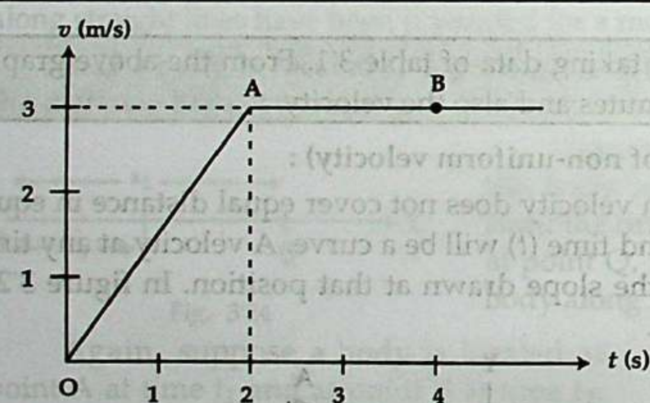


Fig. 3·29(a)

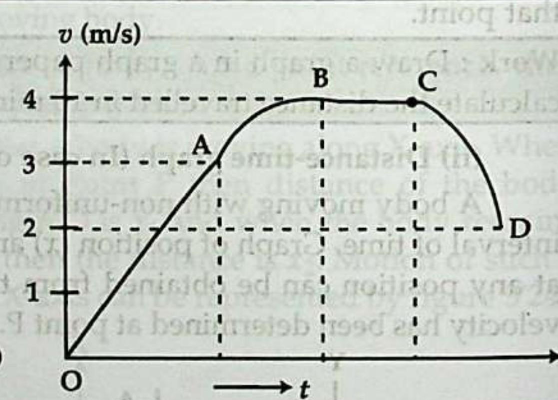


Fig. 3·29(b)

Work exercise II. Draw the above graph 3·29(b) in a graph paper and explain velocities at points A, B, C and D.

(ii) Velocity-time graph (In case of non-uniform velocity) : Decrease of velocity for every 2 seconds starting from $t = 0$ sec is given in table 3·2. If a graph is drawn from the values of $v - t$, the graph will look like the figure 3·30.

From this graph, velocity of the body at any time t can be determined. It is seen from the graph that velocity decreases with time. Besides, it is also seen from the graph that velocity of the body at time 0 is 14 ms^{-1} and becomes zero in 12 sec. It is a non-uniform velocity. Here velocity decreases with time and in every case acceleration (or retardation) remains constant.

Table : 3·2

Time t sec	Velocity v ms^{-1}
0	14
2	12
4	10
6	8
8	6
10	4
12	0

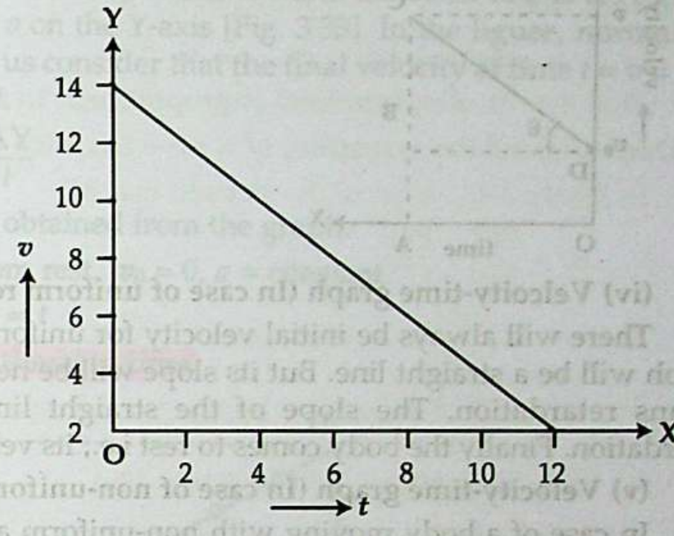


Fig. 3·30

(iii) Velocity-time graph (In case of uniform acceleration) :

The graph of velocity-time of a body moving along a straight line with a uniform velocity will be a straight line. The graph is like this because in equal interval of time increase of velocity becomes same. If the body starts from rest the straight line will pass through the orgin. In Fig. 3·31(a) OB is a straight line. Acceleration can be drawn from the slope of the straight line.

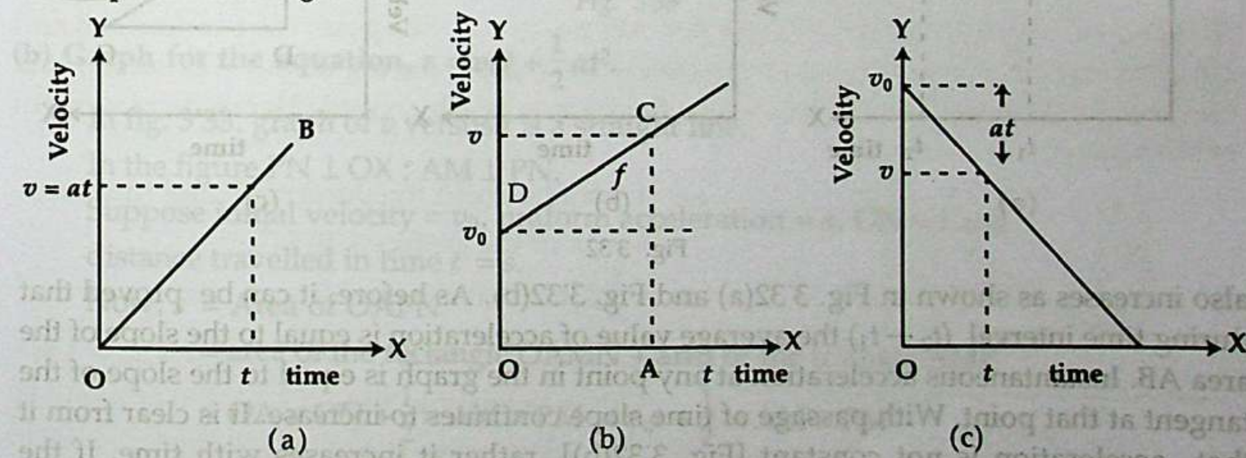
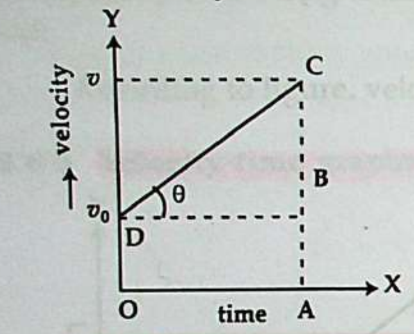


Fig. 3·31

But, if the body has initial velocity, then the velocity-time graph will be a straight line DC [Fig. 3·31(b)]. Here OD = initial velocity v_0. In both the cases the slope of the straight line becomes equal to the uniform acceleration of the body.

Determination of acceleration : In figure 3·31(b) time t = OA, initial velocity, v_0 = OD, final velocity, v = AC



$$\text{acceleration, } a = \frac{\text{change of velocity}}{\text{time}}$$
$$= \frac{AC - OD}{OA} = \frac{AC - AB}{BD}$$
$$= \frac{BC}{DB}$$
= slope of straight line of DC
= tan θ (constant)

(iv) Velcoity-time graph (In case of uniform retardation) :

There will always be initial velocity for uniform retardation. In this case also, the graph will be a straight line. But its slope will be negative [Fig. 3·32(c)]. Negative slope means retardation. The slope of the straight line becomes equal to the uniform retardation. Finally the body comes to rest i.e., its velocity becomes zero.

(v) Velocity-time graph (In case of non-uniform acceleration :

In case of a body moving with non-uniform acceleration the velocity-time graph becomes a curve [Fig. 3·32(a) and 3·32(b)]. Where velocity increases with time, acceleration

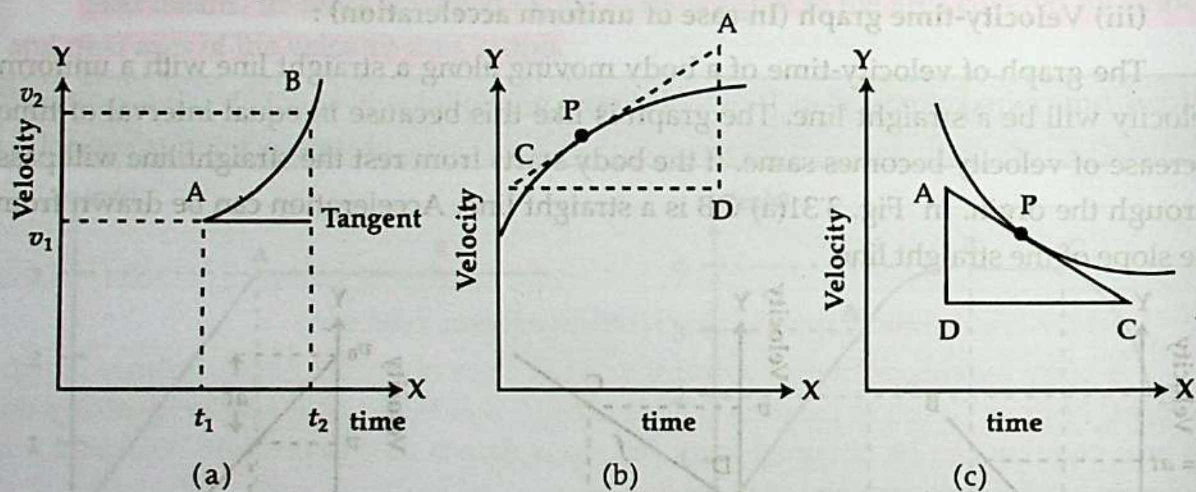


Fig. 3·32

also increases as shown in Fig. 3·32(a) and Fig. 3·32(b). As before, it can be proved that during time interval $(t_2 - t_1)$ the average value of acceleration is equal to the slope of the area AB. Instantaneous acceleration at any point in the graph is equal to the slope of the tangent at that point. With passage of time slope continues to increase. It is clear from it that acceleration is not constant [Fig. 3·32(b)], rather it increases with time. If the velocity of the body decreases with time or if there is retardation, the graph becomes like fig. 3·31(c). Acceleration of the point P is obtained from the slope of ΔADC.

$$\therefore \quad \text{Acceleration} = \text{slope} = \frac{AD}{DC}.$$

3·6·4 Derivation of equations of motion with the help of graphs

(a) Graph in case of the equation, $v = v_0 + at$

In this equation, there are two variables— one is t and the other one is v. A graph is drawn taking t on the X-axis and v on the Y-axis [Fig. 3·33]. In the figure, normal PN is drawn from point P to Y-axis. Let us consider that the final velocity at time $t = v =$ OY. Now OY = OA + AY i.e., $v = v_0 + at$.

Here, slope, $a = \frac{PM}{AM} = \frac{AY}{AM} = \frac{AY}{t}$

$\therefore$ The equation $v = v_0 + at$ is obtained from the graph.

In case of motion of a body from rest, $v_0 = 0$, a = constant.

$\therefore\ v = 0 + \text{constant} \times t \quad \therefore\ v \propto t$

That means, **velocity is proportional to time.**

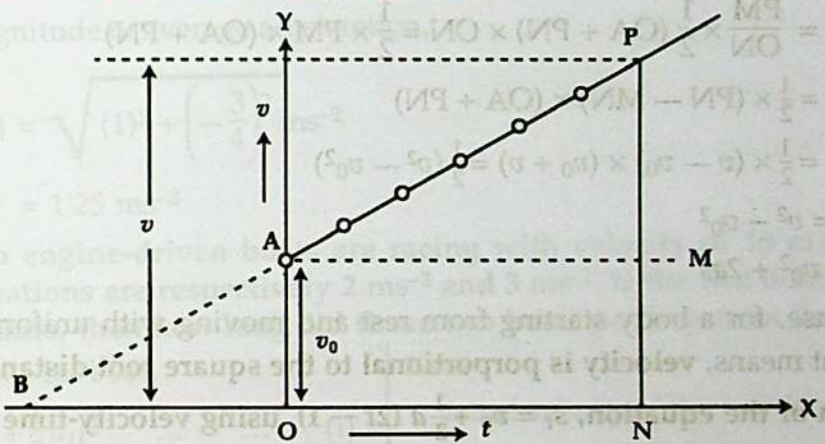


Fig. 3·33

(b) Graph for the equation, $s = v_0t + \frac{1}{2}at^2$.

In fig. 3·33, graph of v versus t is a straight line.

In the figure PN ⊥ OX ; AM ⊥ PN.

Suppose initial velocity = v_0, uniform acceleration = a, ON = t and distance travelled in time $t = s$.

Now, s = Area of OAPN

= area of the rectangle OAMN + area of the triangle AMP

$= \text{OA} \times \text{ON} + \frac{1}{2} \times \text{AM} \times \text{PM} = v_0t + \frac{1}{2} \times \text{AM} \times \text{PM}$

Again, slope $a = \frac{PM}{AM} = \frac{PM}{t}$

$\therefore$ PM = at

$\therefore\ s = v_0t + \frac{1}{2}t \times at = v_0t + \frac{1}{2}at^2$

So, the equation can be presented in graph.

In case of a body moving with uniform acceleration

from rest, $s = 0 \times t + \frac{1}{2} \times \text{constant} \times t^2$

or, $s = \text{constant} \times t^2$ or, $s \propto t^2$

That means, displacement is proportional to time.

(c) Derivation of the equation, $v^2 = v_0^2 + 2as$ with the help of velocity-time graph

In figure 3·33, slope of the straight line AP,

$$a = \frac{PM}{AM} = \frac{PM}{ON}$$

$$\text{So, } as = \frac{PM}{ON} \times \text{area of surface OAPN}$$

$$= \frac{PM}{ON} \times \frac{1}{2}(OA + PN) \times ON = \frac{1}{2} \times PM \times (OA + PN)$$

$$= \frac{1}{2} \times (PN - MN) \times (OA + PN)$$

$$= \frac{1}{2} \times (v - v_0) \times (v_0 + v) = \frac{1}{2}(v^2 - v_0^2)$$

or, $2as = v^2 - v_0^2$

or, $v^2 = v_0^2 + 2as$

In this case, for a body starting from rest and moving with uniform acceleration is $v \propto \sqrt{s}$. That means, **velocity is porportional to the square root distance.**

(d) Derivation of the equation, $s_t = v_0 + \frac{1}{2} a (2t - 1)$ using velocity-time graph

In velocity-time graph 3·34 the straight line AP indicates the motion of a particle. That means, equation for the straight line AP, $v = v_0 + at$. Suppose, the portions AB and AP in that straight line represent respectively the motion during time $(t - 1)$ second and t second. So portion of line BP indicates the motion of tth second. Hence displacement of t th second,

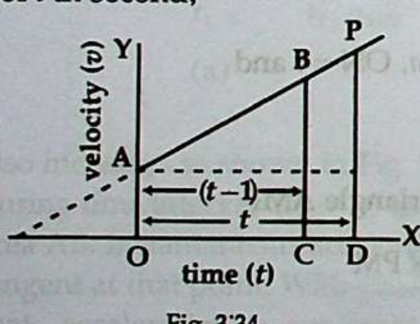


Fig. 3·34

s_t = area between the portion BP and time axis

= area of the surface CBPD

$= \frac{1}{2}(CB + DP) \times CD$

In fig. 3·34, CD $= OD - OC = t - (t - 1) = 1$ s

CB = velocity at time $(t - 1) = v_0 + a(t - 1)$

DP = t velocity at time $t = v_0 + at$

$$\therefore \quad s_t = \frac{1}{2}[v_0 + a(t - 1) + v_0 + at] \times 1$$

$$= \frac{1}{2}[2v_0 + a(2t - 1)]$$

$$= v_0 + \frac{1}{2} a (2t - 1)$$

Mathematical examples

1. The velocity of a body is increased in 8 *s* from $(4\hat{i} + 2\hat{j})$ ms^{-1} to $(12\hat{i} - 4\hat{j})$ ms^{-1}, calculate the average acceleration.

According to the question, $\Delta\vec{v} = \{(12\hat{i} - 4\hat{j}) - (4\hat{i} + 2\hat{j})\}$ ms^{-1}

$= (8\hat{i} - 6\hat{j})$ ms^{-1} and

$\Delta t = 8s$

$\therefore$ average acceleration, $\vec{a} = \frac{\Delta\vec{v}}{\Delta t} = \frac{(8\hat{i} - 6\hat{j})}{8s}$ ms^{-2}

$= (\hat{i} - \frac{3}{4}\hat{j})$ ms^{-2}

and magnitude of average acceleration,

$$|\vec{a}| = \sqrt{(1)^2 + \left(-\frac{3}{4}\right)^2} \text{ ms}^{-2}$$

$= 1{\cdot}25$ ms^{-2}

2. Two engine-driven boats are racing with velocity of 10 ms^{-1} and 5 ms^{-1}. Their accelerations are respectively 2 ms^{-2} and 3 ms^{-2}. If the two boats reach the end at the same time, then how long did those two boats participate in the race ?

Here

Initial velocity of the 1st boat $= v_{01} = 10$ ms^{-1}

Acceleration of the 1st boat, $a_1 = 2$ ms^{-2}

Initial velocity of the 2nd boat $= v_{02} = 5$ ms^{-1}

Acceleration of the 2nd boat, $a_2 = 3$ ms^{-2}

In case of 1st boat,

$$s = v_{01}t + \frac{1}{2}a_1t^2 \quad \ldots \quad \ldots \quad (i)$$

In case of 2nd boat,

$$s = v_{02}t + \frac{1}{2}a_2t^2 \quad \ldots \quad \ldots \quad (ii)$$

From equations (i) and (ii) we get,

$$v_{01}t + \frac{1}{2}a_1t^2 = v_{02}t + \frac{1}{2}a_2t^2$$

$$(v_{01} - v_{02})t = \frac{1}{2}(a_2 - a_1)t^2 \quad \ldots \quad \ldots \quad \ldots \quad (iii)$$

or, $(10 - 5)t = \frac{1}{2}(3 - 2)t^2$

or, $5t = \frac{1}{2} \times t^2$

or, $5 = \frac{1}{2}t$

$\therefore$ $t = 10$ sec

3. A train starts from rest with acceleration $10\ ms^{-2}$. Parallel to this a car starts at the same time with uniform speed $100\ ms^{-1}$. When will the train overtake the car ?

Here,

Initial velocity of the train, $v_{0_1} = 0$

Acceleration of the train, $a_1 = 10\ ms^{-2}$

Initial velocity of the car, $v_{0_2} = 100\ ms^{-1}$

Acceleration of the car, $a_2 = 0$

Time, $t = ?$

Now, for train

$$s = v_{0_1}t = \frac{1}{2}a_1t^2$$

$$\text{or,} \quad s = 0 + \frac{1}{2} \times 10 \times t^2$$

$$\text{or,} \quad s = 5t^2 \quad \dots \quad \dots \quad \text{(i)}$$

and for car

$$s = v_{0_2}t + \frac{1}{2}a_2t^2$$

$$\text{or,} \quad s = 100t + \frac{1}{2} \times 0$$

$$\text{or,} \quad s = 100t \quad \dots \quad \dots \quad \text{(ii)}$$

From (i) and (ii) we get,

$$5t^2 = 100t$$

$$\therefore \quad t = \frac{100}{5} = 20\ s$$

Alternative method : In case of uniform velocity,

$$s = vt = 100\ t$$

$$\therefore \quad t = \frac{100}{s} = \frac{100}{5} = 20\ s$$

4. A bullet after penetrating 0·04 m of a wall loses 75% of its speed. How far will the bullet penetrate that wall afterwards ?

[R. B. 2010; D. B. 2007, 2001; J. B. 2000]

Here, in the first case,

initial velocity $v_{x0} = v_{x0}$

distance, $s_1 = 0·04$ m

final velocity after penetrating 0·04 m,

$$v_{x_1} = (v_{x0} - 75\% v_{x0})$$

$$= (v_{x0} - \frac{3}{4}v_{x0}) = \frac{v_{x0}}{4}$$

acceleration, $a_x = ?$

We know,

$$v_{x_1}^2 = v_{x0}^2 + 2a_xs_1$$

$$\text{or,} \quad a_x = \frac{v_{x_1}^2 - v_{x0}^2}{2s_1} = \frac{\left(\frac{v_{x0}}{4}\right)^2 - v_{x0}^2}{2s_1}$$

$$= \frac{\frac{v_{x0}^2}{16} - v_{x0}^2}{2s_1} = \frac{-15v_{x0}^2}{32s_1}$$

$$= \frac{-15v_{x0}^2}{32 \times 0·04} = \frac{-15v_{x0}^2}{1·28}$$

Again, $v_{x_2}{}^2 = v^2{}_{x_1} + 2a_x s_2$

or, $0 = \frac{vx_0^2}{16} + 2 \times \left(\frac{-15vx_0^2}{1\cdot28}\right) \times s_2$

$= \frac{v_{x0}^2}{16} + \frac{30v_{x0}^2}{1\cdot28} s_2$

$= \frac{30v_{x0}^2}{1\cdot28} s_2 = \frac{v_{x0}^2}{16}$

$\therefore \quad s_2 = \frac{1\cdot28}{480} = 2\cdot67 \times 10^{-3}$ m

Here, in the second case,

initial velocity, $v_{x_1} = \frac{v_{x_0}}{4}$

acceleration, $a_x = -\frac{15\, v_{x_0}^2}{1\cdot28}$

final velocity, $v_{x_2} = 0$

distance, $s_2 = ?$

3·7 Projectile motion

If you go to a stadium to enjoy cricket game then you will see that the cricket ball thrown from the boundary is seen to be ascending first and then descending to the ground in a curved path. Similar phenomena are observed in case of motion of bullet fired upward from a gun, motion of arrows, bomb thrown from plane— in all cases same type of trajectories are observed. This type of curved motion is called **projectile motion** and the path is called **trajectory**. It is a parabola. This type of motion is two dimensional motion. If we ignore air-resistance, motion of a projectile is due to gravity only. The path of projectile is always parabolic. When the projectile reaches the maximum height, its velocity becomes minimum. Again, the motion of the projectile at maximum height is one dimensional. Maximum distance that a projectile travels horizontally is called the **range** of the projectile. Acceleration along horizontal, $a_x = 0$, and vertical acceleration , $a_y = -g$. At the point of ejection the coordinate of the origin is $x = 0, y = 0$. Suppose a projectile is thrown from point O vertically at an angle of θ with initial velocity v_0 [Fig. 3·35]. Initial velocity of the projectile v_0 can be resolved into two components. One component along OX and the other one along OY. The two components are,

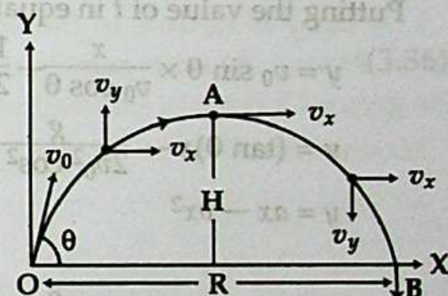


Fig. 3·35

$$v_{x_0} = v_0 \cos\theta \text{ and } v_{y_0} = v_0 \sin\theta.$$

It is to be mentioned that as there is no acceleration along the horizontal direction hence component of velocity along the horizontal direction remains constant. At the maximum height the motion of a projective becomes one dimensional and vertical component of velocity becomes zero. Since there is acceleration along the vertical direction hence component of velocity changes.

We can count time from the instant of ejection i.e., if distance travelled in time $t = 0$ is x, then according to the equation of motion, $x = v_{x_0}t + \frac{1}{2}a_x t^2$

$$x = v_0 \cos \theta t + 0 \quad [\because \text{ Horizontal motion } v_{x0} = v_0 \cos \theta]$$

$$\therefore \quad t = \frac{x}{v_0 \cos \theta} \quad \ldots \quad \ldots \quad \ldots \quad (3.32)$$

Again, as $a_x = -g$, so after time t this velocity of the projectile in the vertical direction,

$$v_y = v_{y_0} - gt = v_0 \sin \theta - gt \quad \ldots \quad \ldots \quad \ldots \quad (3.33)$$

If after time t, the projectile reaches to a height y, then

$$y = v_{y0}t - \frac{1}{2}gt^2 = v_0 \sin \theta \, t - \frac{1}{2}gt^2 \quad \ldots \quad \ldots \quad (3.34)$$

Resultant velocity at time t, $v = \sqrt{v_x^2 + v_y^2}$

If the resultant velocity makes an angle α with the horizontal, then $\tan \alpha = \frac{v_y}{v_x}$

Putting the value of t in equation (3.34) we get,

$$y = v_0 \sin \theta \times \frac{x}{v_0 \cos \theta} - \frac{1}{2} g \times \frac{x^2}{v_0^2 \cos^2 \theta}$$

$$y = (\tan \theta)x - \frac{g}{2v_0^2 \cos^2 \theta} x^2$$

$$y = ax - bx^2 \quad \ldots \quad \ldots \quad \ldots \quad (3.35)$$

This is the equation of a parabola.

Here $a = \tan \theta$, $b = \frac{g}{2v_0^2 \cos^2 \theta}$

These two quantities are constant in the path of ejection. So, trajectory of a projectile is a parabola.

Inquisitive work : Why does the horizontal distance of a body ejected vertically upward becomes zero.

Mathematical examples

1. A cannon-shell is fired with velocity of 40 ms^{-1} towards an enemy plane at an angle of 30° with the horizontal. At what height the shell will strike an wall at 30 m away ?

Suppose the shell strikes the wall at a height h.

Here,

Angle of ejection, $\theta = 30°$

Initial velocity, $v_0 = 40$ ms^{-1}

Distance, $s = 30$ m

Height, $y = ?$

Now, horizontal component of the initial velocity,

$$v_{x0} = v_0 \cos 30°$$

$$= 40 \cos 30°$$

$$= 34{\cdot}641 \text{ ms}^{-1}$$

As there is no horizontal component at acceleration, so velocity component will remain constant. Suppose after t sec the shell strikes the wall at a distance of 30 m.

$\therefore \quad v_{x0}t = s$

or, $v_{x0}t = 30$

$\therefore \quad t = \frac{30}{v_{x0}} = \frac{30}{34{\cdot}641} = 8{\cdot}66$ sec

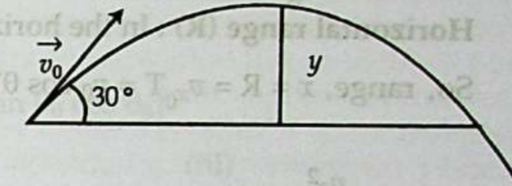


Fig. 3·36

Vertical component of the initial velocity,

$v_{y0} = 40 \sin 30° = 20 \text{ ms}^{-1}$

Vertical displacement after time t,

$$y = v_{y0}t - \frac{1}{2}gt^2 = 20 \times 0{\cdot}866 - \frac{1}{2} \times 9{\cdot}8 \times ({\cdot}866)^2 = 13{\cdot}645 \text{ m}$$

N. B. If the distance along X and Y axies are used together, then by using the law, $y = (\tan\theta)x - \frac{9x^2}{2v_0^2\cos^2\theta}$ the problem can be solved.

Maximum height (H) : At the height point A the vertical component of velocity becomes zero, i.e., $v_y = 0$.

Now from equation (3.33) we get,

$$v_0 \sin\theta - gt = 0,\ t = \frac{v_0 \sin\theta}{g} \quad \ldots \quad \ldots \quad \ldots \quad (3.36)$$

Putting $y = \text{H}$ in equation (3.34) we get,

$$\text{H} = v_0 \sin\theta \times \left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2}\,\frac{gv_0^2 \sin^2\theta}{g^2}$$

$$\text{H} = \frac{v_0^2 \sin^2\theta}{g} - \frac{v_0^2 \sin^2\theta}{2g}$$

$$= \frac{v_0^2 \sin^2\theta}{g}\left(1 - \frac{1}{2}\right)$$

$$= \frac{v_0^2 \sin^2\theta}{2g}$$

$$\therefore \quad \text{H} = \frac{v_0^2 \sin^2\theta}{2g} \quad \ldots \quad \ldots \quad \ldots \quad (3.37)$$

Time to reach maximum height : At maximum height $v_y = 0$ and if $t = t_m$, then inserting these values in the equation, $v_y = v_0 \sin\theta - gt$, we get, $0 = v_0 \sin\theta - gt_m$ or, $t_m = \frac{v_0 \sin\theta}{g}$.

Time of flight (T) : In this case $y = 0$ for ascending and descending, so from equation (3.34) we get, $v_0 \sin\theta t - \frac{1}{2}gt^2 = 0$

$$\therefore \quad t\left(v_0 \sin\theta - \frac{1}{2}gt\right) = 0 \text{ or, } t = 0$$

$$\therefore \quad t = \frac{2v_0 \sin\theta}{g}$$

$t = 0$ indicates the initial state 0 of the projectile.

So, putting time of flight, t = T we get

$$T = \frac{2v_0 \sin \theta}{g} \quad \dots \quad \dots \quad \dots \quad (3.38)$$

Horizontal range (R) : In the horizontal direction OB = range = R

So, range, $x = R = v_{x_0}T = v_0 \cos \theta T = v_0 \cos \theta \times \frac{2v_0 \sin \theta}{g} = \frac{2v_0^2 \sin \theta \cos \theta}{g}$

$[\because 2 \sin \theta \cos \theta = \sin 2\theta]$

$$\text{or,} \quad R = \frac{v_0^2}{g} \sin 2\theta \quad \dots \quad \dots \quad \dots \quad (3.39)$$

Maximum range (R_{max}) :

For any value of v_0, R becomes maximum, when

$\sin 2\theta = 1$ or, $2\theta = 90°$ or, $\theta = 45°$.

$$\therefore \quad R_{max} = \frac{v_0^2 \sin}{g} 90° = \frac{v_0^2}{g}$$

i.e., the range will be maximum if the projectile is ejected at an angle of 45°.

Mathematical examples

1. A body is thrown in the space making an angle of 60° with the ground at a speed of 49 ms^{-1}. At what maximum height will it ascend ? After how long will it fall on the ground ? What is its horizontal range ?

Here,
velocity of projection, $v_0 = 49$ ms^{-1}
angle of projection, $\theta = 60°$
acceleration due to gravity,
$g = 9.8$ ms^{-2}

Maximum height,

$$H = \frac{v_0^2 \sin^2 \theta}{2g} = \frac{(49)^2 \times (\sin 60°)^2}{2 \times 9.8} = 91.87 \text{ m}$$

Time to reach maximum height,

$$t_m = \frac{v_0 \sin \theta}{g} = \frac{49 \times \sin 60°}{9.8} = 4.33 \text{ sec}$$

If the descending time is T, then

$$T = \frac{2v_0 \sin \theta}{g} = \frac{2 \times 49 \times \sin 60°}{9.8} = 8.66 \text{ s}$$

$$\text{Range, } R = \frac{v_0^2 \sin 2\theta}{g} = \frac{(49)^2 \times \sin (2 \times 60°)}{9.8} = 212.18 \text{ m}$$

2. The horizontal range of a projectile is 79.53 m and time period is 5.3 sec. Find the velocity of projection and angle of projection. [D. B. 2010; Ch. B. 2009, 2004]

Here, R = 79.53 m
T = 5.3 sec
$g = 9.8$ ms^{-2}
v_0 = ?
θ_0 = ?

We know,

$$T = \frac{2v_0 \sin \theta_0}{g} \quad \dots \quad \dots \quad \dots \quad \text{(i)}$$

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{v_0^2 \, 2\sin \theta_0 \cos \theta_0}{g} \quad \dots \quad \dots \quad \text{(ii)}$$

From equations (i) and (ii), we get,

$$\frac{T}{R} = \frac{\dfrac{2v_0 \sin \theta_0}{g}}{\dfrac{v_0^2 \sin 2\theta_0}{g}}$$

$$\therefore \quad \frac{5.3}{79.53} = \frac{1}{v_0 \cos \theta_0} \quad [\because \sin 2\theta_0 = 2 \sin \theta_0 \cos \theta_0]$$

or, $v_0 \cos \theta_0 = 15.006$ ... ... (iii)

Now, from equation (i), $T = \dfrac{2 v_0 \sin \theta_0}{9.8}$

$$\therefore \quad v_0 \sin \theta_0 = \frac{9.8}{2} \times T = \frac{9.8 \times 5.3}{2} = 25.97 \ldots \quad \ldots \quad (iv)$$

From equations (iii) and (iv), we get,

$$\tan \theta_0 = \frac{25.97}{15.006} = 1.7306$$

$\therefore \ \theta_0 = 60°$

Substituting this value of θ in equation (iii), we get,

$v_0 \cos 60° = 15.006$

$v_0 \times 0.5 = 15.006$

$$\therefore \quad v_0 = \frac{15.006}{0.5} = 30 \text{ ms}^{-1}$$

3. A body is ejected from a pillar at a height of 30 m at an angle of 30° with the horizontal with velocity 20 ms^{-1}. Calculate the time of flight of the body. [B. B. 2010]

We know,

$$y - y_0 = v_0 \sin \theta t - \frac{1}{2} g t^2$$

or, $-30 = (20 \times \sin 30)\, t - \dfrac{1}{2} \times 9.8 \times t^2$

or, $4.9t^2 - 10\,t - 30 = 0$

or, $t = \dfrac{-(-10) \pm \sqrt{(10)^2 - 4 \times 4.9 \times (-30)}}{2 \times 4.9} = \dfrac{+10 \pm 26.2}{9.8}$

$\therefore \ t = 3.67$ sec or, $t = -1.63$ sec

negative value of t is not acceptable

$\therefore \ t = 3.67$ sec

4. A bullet is thrown from the ground at an angle of 30° with velocity 40 ms^{-1}. At what height will the bullet hit a wall at a distance of 30 m.

We know,

$$y = x \tan \theta - \frac{g x^2}{2 v_0^2 \cos^2 \theta}$$

$$= 30 \times \tan 30° - \frac{9.8 \times (30)^2}{2 \times (40)^2 \times \cos^2 30°}$$

$= 17.32 - 3.68 = 13.64$ m

Here,

$x = 30$ m

$v_0 = 40$ ms^{-1}

$\theta = 30°$

3·8 Equation of motion of a horizontal projectile

Let an object be thrown from a point O with velocity v_0 along the horizontal direction [Fig. 3·37]. If air resistance and the variation of g with height are neglected then horizontal velocity at any point along the motion of the ejected body will be identical and it will be v_0. But as there is no vertical component of the velocity of the ejected body, so downward velocity of the body due to acceleration because of gravity will increase proportionally to time. Let after t sec the object reaches at point P travelling distances x and y respectively along horizontal and vertical directions. Now let the velocity at P be v. The horizontal and vertical components of v are respectively v_x and v_y. Then

$$v_x = v_0 = v \cos\theta \text{ and}$$

$$v_y = v_0 + gt = 0 + gt = v \sin\theta \quad [\because v_0 = 0 \text{ along vertical direction}]$$

$$\therefore \quad v = \sqrt{v_x^2 + v_y^2}$$

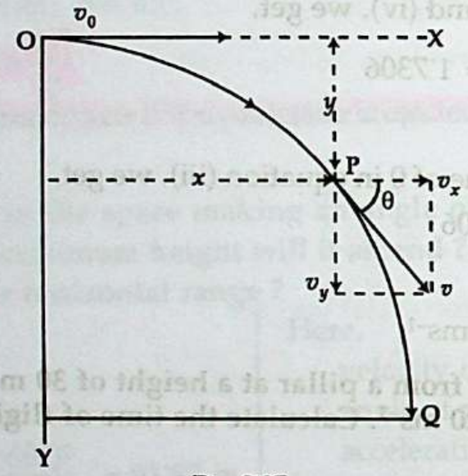


Fig. 3·37

Here θ is the angular separation between the horizontal component v_x and v.

$$\therefore \quad \tan\theta = \frac{v_y}{v_x}$$

Again, $x = v_0 \times t$ ... ... (3.40) [$\because a_x = 0$, along horizontal direction]

$$\therefore \quad t = \frac{x}{v_0}$$

and $y = \frac{1}{2} gt^2$ ... ... (3.41) [$\because v_0 = 0$, along vertical direction]

Now inserting the value of t from equation (3.40) in equation (3.41) we get,

$$y = \frac{1}{2} g \left(\frac{x}{v_0}\right)^2$$

$$\therefore \quad x^2 = \frac{2v_0^2}{g} y \quad \text{... ... ... (3.42)}$$

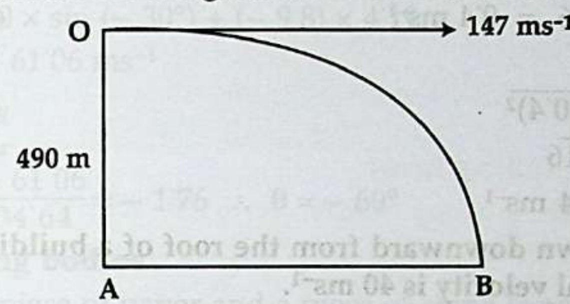
Since v_0 and g are constant, if we put $\frac{2v_0^2}{g} = 4A$ in the above equation we get,

$$x^2 = 4Ay \quad \text{... ... ... (3.43)}$$

This is an equation of a parabola. So, trajectory of a body or a projectile ejected horizontally in a resistance-free path forms a parabola.

Mathematical examples

1. A plane is flying at a height of 490 m from the ground with a horizontal velocity of 147 ms^{-1}. It dropped an object vertically down to a point A on the ground. The object strikes at point B on the ground. What is the distance of AB ?



Suppose, the object is dropped from position O; so, OA = 490 m, after time t the object strikes the ground at point B. Initial velocity of the object in the vertical direction is zero.

Alternative :

$$y - y_0 = v_{y0}t + \frac{1}{2}gt^2$$

or, $-490 = 0 - \frac{1}{2} \times 9.8 \times t^2$

or, $490 = 4.9\ t^2$

$\therefore\ t = 10$ sec

$\therefore\ h = v_{y0}\,t + \frac{1}{2}gt^2$

or, $490 = 0 + \frac{1}{2} \times 9.8t^2$

or, $490 = 4.9\,t^2$

or, $t^2 = 100$

$\therefore\ t = 10$ sec

Here

$h = 490$ m

$v_{x0} = 147$ m

$v_{y0} = 0$

$\therefore$ Horizontal displacement, $x = v_{x0} \times t = 147 \times 10 = 1470$ m

2. A football is kicked making an angle 30° with the ground at velocity 40 ms^{-1}. Find the magnitude of velocity of the football after 2 sec.

[R. B. 2010, 2007; D. B. 2006]

Let the point from where the ball is kicked is the origin and Y-axis is positive vertically upward.

If v_x and v_y are respectively the horizontal and vertical components of final velocity v,

$$\therefore\ v = \sqrt{v_x^2 + v_y^2}$$

Here,

Angle of projection, $\theta = 30°$

Initial velocity, $v = 40$ ms^{-1}

Time, $t = 2$ sec

Final velocity, $v = ?$

If horizontal and vertical components of initial velocity v_0 are v_{x0} and v_{y0} respectively, we get,

$v_x = v_{x_0} + a_x t$

$= v_0 \cos\theta + a_x t$

$= v_0 \cos\theta$ [$\because$ horizontal acceleration, $a_x = 0$]

$= 40 \cos 30°$

$= 34.64$ ms^{-1}

And, $v_y = v_{y_0} + a_y t$

$= v_0 \sin\theta + a_y t$

or, $v_y = 40 \sin 30° + (-9{\cdot}8) \times 2$ [As the vertical component is downward, so

$= 20 - 19{\cdot}6 = 0{\cdot}4 \text{ ms}^{-1}$ $a_y = -g = -9{\cdot}8 \text{ ms}^{-2}$]

$\therefore v = \sqrt{v_x^2 + v_y^2}$

$= \sqrt{(34{\cdot}64)^2 + (0{\cdot}4)^2}$

$= \sqrt{1199{\cdot}9 + 0{\cdot}16}$

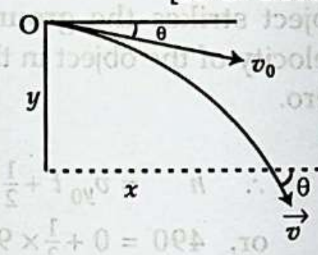
$= \sqrt{1200} = 34{\cdot}64 \text{ ms}^{-1}$

3. A body is thrown downward from the roof of a building of height 170 m at an angle of 30°. Its initial velocity is 40 ms^{-1}.

(a) How long will it take to strike the ground ?

(b) How far from the bottom point of the building it will strike the ground ?

(c) At what angle will it strike the ground ? [R. B. 2005]



Let the ejection point is the origin and vertical Y-axis be positive.

Here, $x_0 = y_0 = 0$

vertical displacement $(y - y_0) = -170$ m (downward)

Angle of ejection, $\theta_0 = -30°$ (the angle is downward along the horizontal)

(a) $t = ?$ (b) $(x - x_0) = ?$ (c) $\theta = ?$

Here horizontal angle $a_x = 0$, vertical acceleration $a_y = -g = -9{\cdot}8 \text{ ms}^{-2}$

(a) We know,

$$y - y_0 = v_{y_0} + \frac{1}{2} a_y t^2 = v_0 \sin\theta_0 t + \frac{1}{2} a_y t^2$$

or, $-170 = 40 \sin(-30°) \times t - \frac{1}{2} \times 9{\cdot}8 \times t^2$

or, $-170 = -20t - 4{\cdot}9t^2$

or, $4{\cdot}9t^2 + 20t - 170 = 0$

or, $t = \dfrac{-20 \pm \sqrt{(20)^2 - 4 \times 4{\cdot}9 \times 170}}{2 \times 4{\cdot}9}$

or, $t = 4{\cdot}9$ sec or, $t = -8{\cdot}27$ sec

so, $t = 4{\cdot}9$ sec

(b) $x = x_0 + v_{x_0} t + \frac{1}{2} a_x t^2$

or, $x - x_0 = v_{x0} t + \frac{1}{2} a_x t^2$

$= v_0 \cos\theta_0 t + \frac{1}{2} a_x t^2$

$= 40 \times \cos(-30°)\, t + \frac{1}{2} a_x t^2$

$= 40 \times \cos 30° \times 4{\cdot}9 + 0$ [$\because \cos(-30°) = \cos 30°$]

$= 40 \times \frac{1}{2} \times 4{\cdot}9 = 145{\cdot}15$ m

(c) $v_x = v_{x0} + a_x t$

$= v_0 \cos\theta_0 + a_x t = 40 \times \cos(-30°) + 0 = 34{\cdot}64\ ms^{-1}$

again, $v_y = v_{y0} + a_y t = v_0 \sin\theta_0 + a_y t$

$= 40 \times \sin(-30°) + (-9{\cdot}8) \times 4{\cdot}19$

$= -61{\cdot}06\ ms^{-1}$

$$\tan\theta = \frac{v_y}{v_x}$$

or, $\tan\theta = \dfrac{-61{\cdot}06}{34{\cdot}64} = -1{\cdot}76 \therefore \theta = -60°$

3·9 Laws of falling bodies

If you drop a piece of paper and a piece of stone from the roof at the same time, what will you see ? You will see that the piece of stone reaches the ground earlier than the piece of paper.

We know the reason behind the falling of the stone or paper is due to gravitational attraction. As acceleration due to gravity does not depend on mass then why the piece of paper and stone do not reach the ground at the same time ? Here, in this case air resistance is responsible for the delay of falling the paper piece. Famous Italian Scientist Galileo Galili conducted extensive research on motion of falling bodies and gave some experimental laws. In 1589, he allowed to fall different heavy bodies from the top of an inclined pillar of 180 ft height and showed that these bodies fall on the ground almost at the same time [Fig. 3·38]. A small difference of delay in falling of the light and heavy bodies is due to air resistance. Later on Scientist Newton proved this law by the famous Guinea and Feather experiment. Galileo gave three laws relating freely falling bodies like these. These laws are :


Fig. 3·38

First law : In vacuum, all the freely falling bodies starting from rest traverse equal distance in equal interval of time. Or in vacuum all bodies starting from rest fall with equal rapidity.

Explanation : Suppose two bodies having masses m_1 and m_2 fall downward from a long height without any hindrance and cover distances h_1 and h_2 in time t. Now according to this law, it will be found that $h_1 = h_2$.

Second law : Starting from rest, the velocity of a freely falling body is propotional to the time taken to fall.

Explanation : Suppose a body is falling to the ground from a great height under the action of gravitational force. In this case, if the velocity of the body is v, then $v \propto t$.

or, $\dfrac{v}{t} =$ constant or, $\dfrac{v_1}{t_1} = \dfrac{v_2}{t_2} =$ constant

Here v_1 and v_2 are respectively the velocities in time t_1 and t_2.

Third law : Starting from rest, the distance traversed by a freely falling body is proportional to the square of the time to fall.

Explanation : Suppose a body is falling freely to the ground from a great height. If the distance traversed in time t is h, then according to this law $h \propto t^2$.

$$\frac{h}{t^2} = \text{constant} \quad \text{or,} \quad \frac{h_1}{t_1^2} = \frac{h_2}{t_2^2} = \text{constant}$$

Here h_1 and h_2 are respectively the distance traversed at time t_1 and t_2 .

Equation of motion of a falling body

Suppose a body is falling along Y-axis, vertically with initial velocity v_0 freely under the action of gravity from a height h. Now in case of falling body,

$$v = v_0 + gt \quad \ldots \quad \ldots \quad \ldots \quad (3.44)$$

$$h = v_0 t + \frac{1}{2} g t^2 \quad \ldots \quad \ldots \quad \ldots \quad (3.45)$$

and $v^2 = v_0^2 + 2gh \quad \ldots \quad \ldots \quad \ldots \quad (3.46)$

Initial velocity for the freely falling body, $v_0 = 0$, so

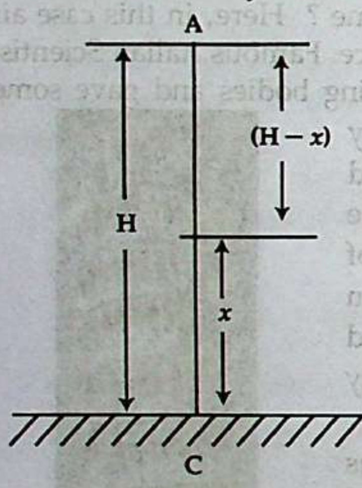


Fig. 3·39

$$v = gt$$

$$h = \frac{1}{2} g t^2$$

and $v^2 = 2gh$

To determine time to reach maximum height :

Let at time $t = T_1$, the body reaches the maximum height $y = H$ [Fig. 3·39]. There the final velocity, $v = 0$. In this case, using the equation $v = v_0 - gt$ we get

$$0 = v_0 - gT_1 \quad [\because \text{ at maximum height } v = 0]$$

or, $T_1 = \frac{v_0}{g} \quad \ldots \quad \ldots \quad (3.47)$

It indicates the time of descend.

To determine time taken for travelling up and down :

Suppose the body at time $t = T_2$ reaches the maximum height and comes back to the initial position $y = 0$.

In this situation

$$y = v_0 t - \frac{1}{2} g t^2$$

or, $0 = v_0 T_2 - \frac{1}{2} g T_2^2$

or, $\frac{1}{2} g T_2^2 = v_0 T_2$

or, $T_2 = \frac{2v_0}{g} \quad \ldots \quad \ldots \quad \ldots \quad (3.48)$

Ascending time,

$$T_0 = T_2 - T_1 = \frac{2v_0}{g} - \frac{v_0}{g} = \frac{v_0}{g} \quad \ldots \quad \ldots \quad \ldots \text{(d)} \quad (3.49)$$

So, it is seen that ascending and descending times are equal to each other.

Work for self-practice : Throw a piece of paper from the roof and ask your friend to drop a mango from a tree. Will the acceleration be uniform in both the cases ? Find out the cause of it.

3·10 Uniform circular motion

If you hold one end of a thread that has a stone fastened at the other end and if you rotate the thread over your head, you will see that the stone is rotating in a circular path. Now, if you just release the thread, you will never see that the stone after whirling many times has fallen on your head. Rather you will see that the stone has left your hand along the tangent at that point of a circular path from where it has left the hand as in fig. 3·40(a). If you look at the figure below you will understand it clearly. While the stone was whirling over the head, if the magnitude of the velocity was constant at every moment then that type of motion becomes uniform circular motion.

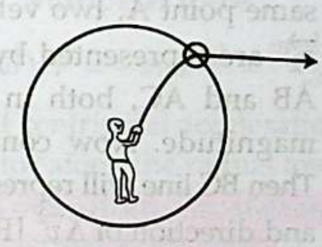


Fig. 3·40 (a)

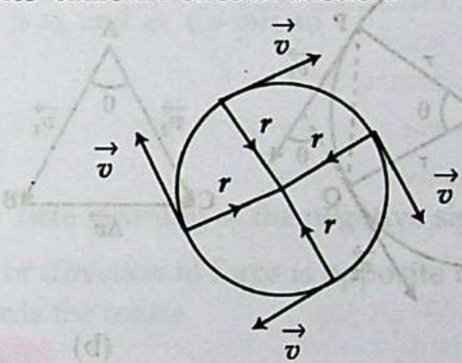


Fig. 3·40 (b)

That means, **if a particle rotates around a circular path at constant speed then that motion is called uniform circular motion.**

In uniform circular motion although the magnitude of velocity of the body remains unchanged, direction of velocity changes. So, velocity changes. The direction of the velocity at any point in a circular path is equal to the tangent at that point. So, a force acts in order to change the direction of velocity i.e., an acceleration acts. The direction of this acceleration is along the normal to the direction of motion and towards the centre of the circle. This acceleration is centripetal acceleration. If the acceleration would have been in different direction, then along the tangent of the circle i.e., along the direction of motion of the particle there would have a component of acceleration; consequently change of speed of the particle would occur.

Why a body travelling in a circular path with constant speed has acceleration ?

In answer we can say, a body moves along the periphery of a circle in uniform speed then that motion of the body is uniform circular motion. Although the body in this motion rotates with constant speed, but direction at every point changes. Tangent drawn at any point along the periphery will be the direction of velocity at that point [Fig. 3·40(b)]. As the directions of tangents at different points are different, so the direction of velocity always changes. That means velocity is also changing, thus there is acceleration. So,it can be said that **a body travelling in circular path with constant speed has acceleration.**

Since speed is constant, so acceleration is always towards the centre. It can be said, **when an object rotates in a circular path then the acceleration that acts along the radius of the circle and towards the centre is called centripetal acceleration.**

Explanation of magnitude and direction of centripetal acceleration

Let a particle moving with velocity v in a circular path PQR, O being the centre and r being the radius of that circle, reach to position P at time t and to position Q at time $(t + \Delta t)$ and subtend to an angle $\angle POQ = \theta$ [Fig. 3·41(a)]. So, at time Δt, the distance travelled, $\Delta s = v\Delta t$ = arc PQ. Instantaneous velocity at points P and Q are respectively v_1 and v_2 and their directions will be along the tangents drawn at those points. The magnitude of both the velocities (v) is same but direction is different. If the change of velocity ($\vec{v_2} - \vec{v_1}$) in time Δt is represented as Δv, the value of Δv can be obtained from the law of triangle of vectors. Again, from the same point A, two velocities $\vec{v_1}$ and $\vec{v_2}$ are represented by straight lines AB and AC, both in direction and magnitude. Now connect B and C. Then BC line will represent magnitude and direction of $\Delta\vec{v}$ [Fig. 3·41(b)].

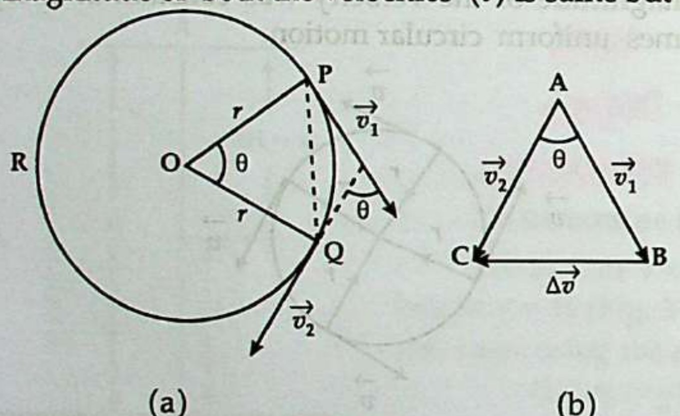


Fig. 3·41

According to description the triangle OPQ formed by OP, OQ and PQ and the triangle ABC are similar. As both of them are isosceles triangles and $\angle BAC = \angle POQ = \theta$. So, if $\angle ABC = \angle ACB = \varphi$, then

$$\varphi = \left(90^\circ - \frac{\theta}{2}\right)$$

Again, according to the principle of similar triangles, $\frac{BC}{AC} = \frac{PQ}{OQ}$

$$\frac{\Delta v}{v} = \frac{v\Delta t}{r} \text{ (approximately)}$$

$$\text{or, } \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

Here the arc PQ has been taken equal to chord PQ.

If Δt is exceedingly small, then the relation can be taken as correct. Since in this condition the arc and the chord are taken almost equal.

When $\Delta t \to 0$, distance between P and Q and angle θ between them will be very small, i.e., two points P and Q will be very close to each other and angle between $\Delta \vec{v}$ and $\vec{v_1}$ or $\vec{v_2}$ will be about $\varphi \approx 90^\circ$ i.e., $\frac{\Delta \vec{v}}{\Delta t}$ will act towards the centre.

So, the magnitude of acceleration,

$$a = \underset{\Delta t \to 0}{\text{Lt}} \left| \frac{\Delta \vec{v}}{\Delta t} \right| = \frac{v^2}{r} \quad \ldots \ldots \ldots \quad (3.50)$$

So, a body moving with uniform velocity of v in a circular path of radius of r, an acceleration, $a = \frac{v^2}{r}$ directed along the centre acts on that body.

Equation (3.49) is the equation of the centripetal acceleration of a body moving in a circular path.

Therefore, if the centripetal force acting on the rotating body is F, then according to the Newton's second law of motion (F = ma)

$$F = \frac{mv^2}{a}.$$

If the angular velocity of the body is ω, and as $v = \omega r$; so

$$F = \frac{mv^2}{r} = \frac{m\omega^2 r^2}{r} = m\omega^2 r$$

Vector form of centripetal force :

$-m(\vec{\omega} \cdot \vec{\omega})\,\vec{r} = -m\omega^2 \vec{r} = \frac{mv^2 \vec{r}}{r^2}$ here meaning of the negative sign (− ve) is that the direction of centripetal acceleration or direction of force is opposite to radius vector or position vector along the radius towards the centre.

Characteristics of uniform circular motion :

1. uniform speed exists in it;
2. uniform angular velocity exists in this motion;
3. its angular acceleration is zero;
4. this motion has angular acceleration.

Work : Explain why centripetal force is needed to rotate a body in uniform circular motion.

According to the first law of motion if no external force is applied on a body, it will remain stationary or in motion with uniform velocity. So to rotate a body in circular path an external force is needed to apply along the perpendicular direction of motion. This external force acting along the radius is towards the centre which is called the centripetal force. So, to rotate a body with uniform circular motion a centripetal force is needed.

Perceptional work : Acceleration does not exist in a body moving along straight line with uniform speed but acceleration exists in a body moving with uniform circular motion—explain.

Mathematical examples

1. The electron of hydrogen atom is revolving around the nucleus in a circular path of radius $5{\cdot}2 \times 10^{-11}$ m with the velocity $2{\cdot}20 \times 10^6$ ms^{-1}. Calculate the centripetal acceleration of the electron.

We know, normal acceleration,

$$a = \frac{v^2}{r} = \frac{(2{\cdot}2 \times 10^6)^2}{5{\cdot}2 \times 10^{-11}}$$
$$= \frac{2{\cdot}2 \times 2{\cdot}2 \times 10^{12} \times 10^{11}}{5{\cdot}2}$$
$$= 9{\cdot}31 \times 10^{22} \text{ ms}^{-2}$$

Here,

Velocity, $v = 2{\cdot}20 \times 10^6$ ms^{-1}

Radius, $r = 5{\cdot}2 \times 10^{-11}$ m

Centripetal acceleration, a = ?

2. A body of mass 50 g is fastened at the end of a thread of length 30 cm and is rotated in a circular path 3 times per second. Calculate the centripetal force.

We know,

$$\omega = 2\pi n \text{ rad s}^{-1} = 2\pi \times 3 \text{ rad s}^{-1}$$
$$= 6\pi \text{ rad s}^{-1}$$

Here,

r = radius of the circle = 30 cm

m = 50 g

n = 3

Again, centripetal acceleration,

$$a = \omega^2 r = (6\pi)^2 \times 30 = (18{\cdot}852)^2 \times 30 = 10662 \text{ cm s}^{-2}$$

Again, centripetal force,

$$F = ma = m\omega^2 r = 50 \times 10662 = 533100 \text{ dyne} \simeq 5{\cdot}33 \text{ N}$$

Alternative method :

$$F = m\omega^2 r = m(2\pi n)^2 \times r$$
$$= 0{\cdot}05\ (2 \times 3{\cdot}14 \times 3)^2 \times 0{\cdot}3$$
$$\simeq 5{\cdot}33 \text{ N}$$

Here,

r = 30 cm = 0·3 m

m = 50 g = 0·05 kg

n = 3

Necessary mathematical formulae

$s = v_0 t \pm \frac{1}{2} a t^2$ ... ... (1)

$s = vt$ (in case of average velocity) ... ... (2)

$v = \frac{ds}{dt}$ ... ... (3)

$a = \frac{dv}{dt}$ ... ... (4)

$v^2 = v_0^2 \pm 2as$ ... ... (5)

$v = gt$ (in case of falling) ... ... (6)

$h = \frac{1}{2} g t^2$ ... ... (7)

$v^2 = 2gh$ (in case of falling) ... ... (8)

$h = \frac{1}{2} g\ (2t - 1)$ ... ... (9)

$v^2 = v_0^2 \pm 2gh$ ... ... (10)

$$h = v_0 t \pm \frac{1}{2} g t^2 \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

$$h = \frac{u^2}{2g} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

$$t = \frac{u}{g} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

$$a = \frac{v^2}{r} = \omega^2 r, \ v = \omega r \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

$$F = \frac{mv^2}{r} = m\omega^2 r \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

$$\omega = 2\pi n = \frac{2\pi}{T} = \frac{2\pi N}{t} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

$$H = \frac{v_0^2 \sin^2 \theta}{2g} \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$$R = \frac{v_0^2 \sin^2 \theta}{g} \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

$$R_{max} = \frac{v_0^2}{g} \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

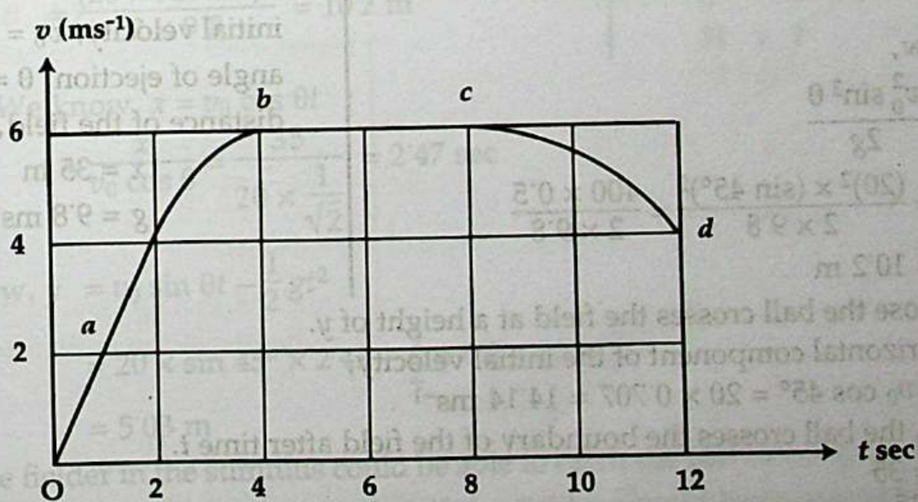
$$T = \frac{2v_0 \sin^2 \theta}{g} \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

$$v = \omega r \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

$$\omega = \frac{2\pi N}{t} \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

Higher efficiency mathematical examples

1. How the change of velocity of Rubel with time during 100 m running competition in the college on Independence day has been shown in the following graph.



(a) How far Rubel has covered between 4 sec and 8 sec ?

(b) Analyse the acceleration of Rubel according to the stimulus.

(a) During 4 sec and 8 sec Rubel was running with uniform velocity.

So, $v = 6\ \text{ms}^{-1}$, $t = (8 - 4) = 4$ sec

$\therefore$ Distance covered $s = vt = 6 \times 4 = 24$ m

(b) Since between 0 and a he ran with uniform velocity, again between a and b he ran with non-uniform velocity, and between b and c velocity is constant, so here acceleration = 0.

Again, between c and d velocity decreases i.e., there is retardation.

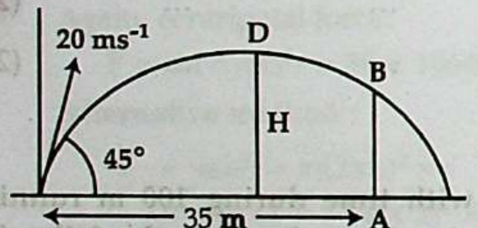
Now, acceleration at point $a = \dfrac{\Delta v}{\Delta t} = \dfrac{4-2}{2-0} = \dfrac{2}{2} = 1\ \text{ms}^{-2}$

acceleration at point $b = 0$

acceleration at point $c = 0$

retardation at point $d = \dfrac{4}{12} = \dfrac{1}{3}\ \text{ms}^{-2}$

2. In a cricket test Tamim struck a ball by a bat and it started moving at 45° angle with velocity 20 ms^{-1} over the head of a bowler. From the middle of the field a fielder of the opposite team started running. The ball reached the viewer-gallery before the fielder could reach the line of ball. The distance travelled by the ball inside the field was 35 m and at that place, g = 9·8 ms^{-2}.



(a) At what height the ball hit by the bat would reach ?

(b) In the stimulus the fielder could catch the ball jumping a height of 3 m. If he would reach the line of ball in right time, could he then catch the ball ? Give mathematical analysis in support of your answer.

(a) Suppose, the ball will reach at a maximum height of H.

We know,

$$H = \frac{v_0^2 \sin^2 \theta}{2g}$$

$$\therefore\ H = \frac{(20)^2 \times (\sin 45°)^2}{2 \times 9{\cdot}8} = \frac{400 \times 0{\cdot}5}{2 \times 9{\cdot}8}$$

$$= 10{\cdot}2 \text{ m}$$

Here,

initial velocity, $v_0 = 20\ \text{ms}^{-1}$

angle of ejection, $\theta = 45°$

distance of the field,

$x = 35$ m

$g = 9{\cdot}8\ \text{ms}^{-2}$

(b) Suppose the ball crosses the field at a height of y.

Now, horizontal component of the initial velocity,

$v_{x0} = v_0 \cos 45° = 20 \times 0{\cdot}707 = 14{\cdot}14\ \text{ms}^{-1}$

Suppose, the ball crosses the boundary of the field after time t.

$\therefore\ v_{x0}t = 35$

$\therefore\ t = \dfrac{35}{14{\cdot}14} = 2{\cdot}475$ s

Vertical component of the initial velocity,

$$v_{y0} = 20 \sin 45° = 20 \times 0.707 = 14.14 \text{ ms}^{-1}$$

Vertical displacement after time t,

$$y = v_{y0}t - \frac{1}{2}gt^2 = 14.14 \times 2.475 - \frac{1}{2} \times 9.8 \times (2.475)^2$$
$$= 35 - 30 = 5 \text{ m}$$

The fielder in the stimulus could jump a height of 3 m, but the ball crossed the field 5 m above, so the fielder could not catch the ball.

3. In a test between Bangladesh and Zimbabwe Sakib struck a ball by a bat and the ball started moving over the baller to outside the field with velocity of 20 ms^{-1} at an angle of 45°. From mid field a fielder started running. Before the fielder reached the line of the ball, it crossed the boundary making six. Distance travelled by the ball inside the field was 35 m. **[D. B. 2015]**

(a) Why the horizontal distance travelled by a body thrown vertically upward is zero ?

(b) At what height did the ball in the stimulus ascent ?

(c) The fielder in the stimulus could catch the ball by jumping 3 m. Could he catch the ball if he would have reached the line of the ball ? Give mathematical analysis in favour of your answer.

(a) Velocity component in the horizontal direction of a body thrown vertically upward is zero. So, horizontal distance is also zero.

(b) Suppose the ball in the stimulus would rise maximum height of H.

We know,

$$H = \frac{(v_0 \sin \theta)^2}{2g}$$
$$= \frac{(20 \times \sin 45°)^2}{2 \times 9.8} = 10.2 \text{ m}$$

Here,

$v_0 = 20 \text{ ms}^{-1}$

$\theta = 45°$

$g = 9.8 \text{ ms}^{-2}$

$H = ?$

(c) We know, $x = v_0 \cos \theta t$

$$t = \frac{x}{v_0 \cos \theta} = \frac{35}{20 \times \frac{1}{\sqrt{2}}} = 2.47 \text{ sec}$$

Now, $y = v_0 \sin \theta t - \frac{1}{2}gt^2$

$$= 20 \times \sin 45° \times 2.47 - \frac{1}{2} \times 9.8 \times (2.47)^2$$
$$= 5.03 \text{ m}$$

The fielder in the stimulus could be able to catch the ball by jumping 3 m height.

Since $y = 5.03 \text{ m} > 3 \text{ m}$, so even if the fielder would have reached the line of the ball, he could not catch the ball.

4. A performer in a circus party is rotating in horizontal plane a sphere weighing 5 kg fastened at the end of a rope of length 2 m and 1·5 m above the ground. The sphere rotates 20 times per minute. While rotating the rope is torn off.

(a) How much force will the rotating sphere experience towards the centre ?

(b) What should be the distance of the observers' row from the performer so that the sphere does not hit any observer ? Explain by mathematical analysis.

(a) We know,

$$F_c = m\omega^2 r = m \times \left(\frac{2\pi N}{t}\right)^2 \times r = 5 \times \left(\frac{2 \times 3{\cdot}14 \times 20}{60}\right)^2 \times 2 = 43{\cdot}87 \text{ N}$$

Here,
$m = 5$ kg
$r = 2$ m
$N = 20$
$t = 1$ min $= 60$ sec
centripetal force, $F_c = ?$

(b) We know,

$$\omega = \frac{2\pi N}{t} = \frac{2 \times 3{\cdot}14 \times 20}{60} = 2{\cdot}0944 \text{ rad s}^{-1}$$

Again, $v = \omega r = 2{\cdot}0944 \times 2 = 4{\cdot}188$ ms^{-1}

If the maximum horizontal range of the sphere is R, then

$$R = \frac{v^2}{g} = \frac{(4{\cdot}188)^2}{9{\cdot}8} = 1{\cdot}79 \text{ m (from the performer)}$$

So, if the distance of the row of observers from the performer is more than 1·79 m, then the sphere will not hit any observer.

5. A hunter fired a bullet at a bird sitting on a wall of 10 m height and 75 m away from the hunter. Angle of ejection of the bullet is 60° and ejection velocity, $v_0 = 30$ ms^{-1}.

(a) Calculate how long the bullet in the stimulus was in air.

(b) Whether the bullet mentioned in the stimulus will hit the bird or not ? Analyse with mathematical justification. **[Ch. B. 2015]**

(a) We know,

$$\text{time of flight, } T = \frac{2v_0 \sin\theta}{g} = \frac{2 \times 30 \times \sin 60°}{9{\cdot}8} = 5{\cdot}302 \text{ s}$$

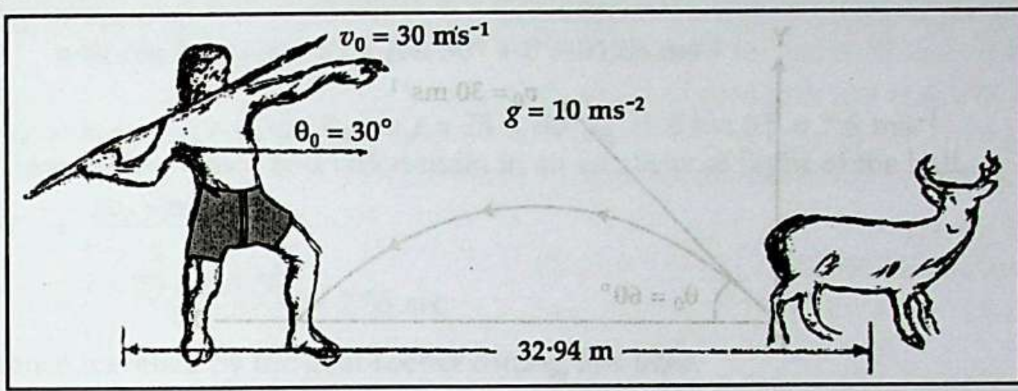
(b) We know,

$$y = (\tan\theta)\, x - \frac{g}{2(v_0 \cos\theta)^2} x^2$$
$$= \tan 60° \times 75 - \frac{9{\cdot}8}{2(30 \times \cos 60°)^2} \times (75)^2$$
$$= 129{\cdot}9 - 122{\cdot}5 = 7{\cdot}4 \text{ m}$$

Here,
$x_0 = y_0 = 0$
$v_0 = 30$ ms^{-1}
$x = 75$ m
$\theta = 60°$
$y = ?$
height of the wall, $h = 10$ m

Since the vertical distance of the bullet is $y = 7{\cdot}4$ m and the position of the bird is on a wall of 10 m height, so the bullet will not hit the bird.

6.



When the hunter throws his spear the deer starts running from rest at a velocity 10 ms^{-1} with uniform acceleration along PQ. **[Ch. B. 2015]**

(a) At what maximum height will the spear of stimulus rise from the point of ejection ?

(b) Will the spear hit the deer ? Give mathematical logic in favour of your answer.

(a) We know, maximum height

$$H = \frac{v_0^2 \sin^2 \theta}{2g}$$

$$= \frac{(30)^2 \times (\sin 30°)^2}{2 \times 10} = 11·25 \text{ m}$$

(b) Horizontal range of the spear,

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

$$= \frac{(30)^2 (\sin 2 \times 30°)}{10} = 77·94 \text{ m}$$

Time of flight of the spear,

$$T = \frac{2 v_0 \sin \theta}{g}$$

$$= \frac{2 \times 30 \times \sin 30°}{10} = 3 \text{ sec}$$

Distance of the deer from the hunter after 3 sec.

$$s = 32·94 + v_0 t + \frac{1}{2} a t^2$$

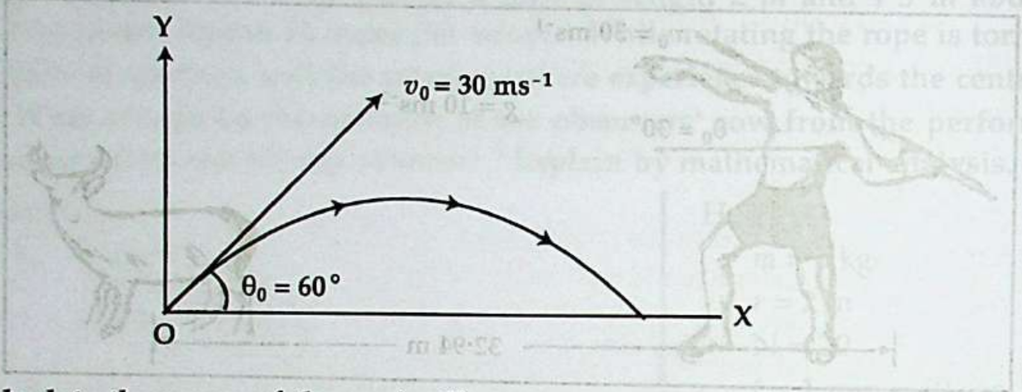
Here, $v_0 = 0$

$$= 32·94 + \frac{1}{2} \times 10 \times (3)^2$$

$$= 77·94 \text{ m}$$

Here the range and the distance travelled by the deer is equal, so the spear will hit the deer.

7.



(a) Calculate the range of the projectile.

(b) Will the projectile cross a wall of height of 25 m at a distance of 20 m from the point of ejection of the projectile along X-axis ? [J. B. 2016]

(a) We know, range of the projectile,

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(30)^2 \times \sin 120°}{9·8}$$

$$= 79·53 \text{ m}$$

(b) If the vertical distance of the projectile at a distance of 20 m is more than 25 m then the projectile will be able to cross the wall and if less, then not.

We know, vertical distance of the projectile,

$$y = \tan \theta_0\, x - \frac{gx^2}{2(v_0 \cos \theta_0)^2}$$

$$y = \tan 60° \times 20 - \frac{9·8 \times (20)^2}{2(30 \times \cos 60°)^2} = 25·93 \text{ m}$$

Since the vertical distance is greater than the height of the wall, so it will be able to cross.

8. A football player kicked a ball from 80 m distance in front of the goal keeper at an angle of 30° with the horizontal with a velocity of 25 ms^{-1}. After some time, in order to catch the ball the goal keeper rushed with uniform velocity of 10 ms^{-1} towards the ball. [g = 9·8 ms^{-2}]

(a) What is the velocity of the ball after 0·5 s of kicking ?

(b) Whether the goal keeper will be able to catch the ball before it touches the ground or not—give opinion with mathematical analysis. [R. B. 2016]

(a) Let the point from where the ball is kicked be the origin and upward direction along Y-axis is positive.

If v_x and v_y are the horizontal and vertical components of the final velocity, then $v = \sqrt{v_x^2 + v_y^2}$

If horizontal and vertical components of the initial velocity are v_{x0} and v_{y0} then,

$v_x = v_{x0} + a_x t$
$= v_0 \cos\theta_0 + a_x t = 25 \times \cos 30° + 0 = 21{\cdot}65\ ms^{-1}$

and

$v_y = v_{y0} + a_y t = v_0 \sin\theta_0 + a_y t = 25 \times \sin 30° - 9{\cdot}8 \times 0{\cdot}5 = 7{\cdot}6\ ms^{-1}$

(b) Time during which ball will remain in air i.e., time of flight of the ball,

$$T = \frac{2v_0 \sin\theta_0}{g} = \frac{2 \times 25 \times \sin 30°}{9{\cdot}8} = 2{\cdot}55 \text{ sec}$$

Distance travelled by the goal keeper during this time,

$s = vt = 10 \times 2{\cdot}55 = 25{\cdot}5$ m

horizontal range of the ball,

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(25)^2 \times \sin 60°}{9{\cdot}8} = 55{\cdot}23 \text{ m}$$

That means, before touching the ground if the goalkeeper can travel at least a distance $(80 - 55{\cdot}23)$ m = $24{\cdot}77$ m, then he will be able to catch the ball. Since the goal keeper travels $25{\cdot}5$ m before the ball touches the ground, so he will be able to catch the ball.

9. During the cricket match between India versus Bangladesh Sakib-Al-Hasan threw the ball towards the batsman Virat Kohli. The bastsman hit the ball at an angle of 30° with velocity of 20 ms^{-1}. Rubel at a distnace of 80 m from the batsman started running with velocity of 8 ms^{-1} in order to catch the ball.

(a) How long will the ball remain in air ?

(b) Will it be possible for Rubel to catch the ball ? Give your decision with mathematical analysis. [B. B. 2016]

(a) We know, time of flight,

$$T = \frac{2v_0 \sin\theta_0}{g} = \frac{2 \times 20 \times \sin 30°}{9{\cdot}8} = 2{\cdot}04 \text{ sec}$$

(b) distance of Rubel from the batsman, $d = 60$ m

Rubel will rush to the ball in $2{\cdot}04$ sec

$\therefore$ distance travelled by Rubel, s_1 = velocity of Rubel × time of flight
$= 8 \times 2{\cdot}04 = 16{\cdot}32$ sec

horizontal range of the ball,

$$R = \frac{v_0^2 \sin 2\theta_0}{g} = \frac{(20)^2 \times \sin 60°}{9{\cdot}8} = 35{\cdot}35 \text{ m}$$

So, disntance travelled by the ball before it touches the ground,

$s_2 = d - R = (60 - 35{\cdot}35)$ m = $24{\cdot}65$ m

That means, Rubel will have to travel a distance of $24{\cdot}65$ m in $2{\cdot}04$ s to catch the ball. But Rubel could travel $16{\cdot}32$ m before the ball touches the ground. Hence it will not be possible to catch the ball.

10. During a football training two players both kicked a football at velocity 10 ms^{-1} making angles 30° and 60° respectively. A goalkeeper was standing to catch the ball just before it drops on the ground.

(a) What is the magnitude of the ball after 1 sec in case of the first player ?

(b) Without changing the position the goalkeeper will be able to catch the two balls at different times—mathematically verify its truth. **[C. B. 2017]**

(a) We know,

horizontal component of the velocity,

$v_x = v_0 \cos\theta = 10 \cos 30°$

$= 10 \times 8{\cdot}66\ ms^{-1} = 8{\cdot}66\ ms^{-1}$

and vertical component,

$v_y = v_0 \sin\theta - gt = 10 \sin 30° - gt$

$= 10 \sin 30° - 9{\cdot}8 \times 1 = 5 - 9{\cdot}8$

$= -4{\cdot}8\ ms^{-1}$

Here,

angle of ejection, $\theta = 30°$

initial velocity, $v_0 = 10\ ms^{-1}$

time, $t = 1$ sec

velocity of the ball, v = ?

$\therefore$ magnitude of the velocity $|\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(8{\cdot}66)^2 + (-4{\cdot}8)^2}$

$= \sqrt{98{\cdot}03} = 9{\cdot}90\ ms^{-1}$

((b) Initial velocity of the first and the second players, $v_0 = 10\ ms^{-1}$

angle of ejection of the first player = 30°

and angle of ejection of the second player = 60°

horizontal component of the ball of the first player,

$$R_1 = \frac{v_0^2 \sin 2\theta_1}{g} = \frac{(10)^2 \sin(2 \times 30°)}{9{\cdot}8} = 8{\cdot}837\ m$$

horizontal component of the ball of the second player,

$$R_2 = \frac{v_0^2 \sin 2\theta_2}{g} = \frac{(10)^2 \times \sin(2 \times 60°)}{9{\cdot}8} = 8{\cdot}837\ m$$

time of flight of the first ball,

$$T_1 = \frac{2v_0 \sin\theta_1}{g} = \frac{2 \times 10 \times \sin 30°}{9{\cdot}8} = 1{\cdot}02\ sec$$

time of flight of the second ball,

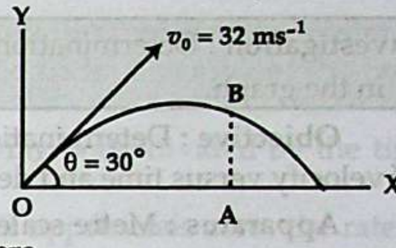
$$T_2 = \frac{2v_0 \sin\theta_2}{g} = \frac{2 \times 10 \times \sin 60°}{9{\cdot}8} = 1{\cdot}767\ sec$$

Therefore, $R_1 = R_2$ but $T_1 \neq T_2$; hence without changing the position the goalkeeper will be able to catch the ball at different times.

11. Two friends Sumon and Rana observed that ejecting a body from a point O on earth with a velocity of 32 ms^{-1} at an angle of 30° drops on earth at a distance of 85 m over a wall AB of height 2 m. **[D. B. 2017]**

(a) Calculate the velocity of the ejected body after 1·2 s of ejection from the point O.

(b) According to the stimulus for what smallest change of angle of ejection the projectile will be obstructed by the wall AB ? Give opinion with mathematical analysis.



Here,
angle of ejection, θ = 30°
initial velocity, v_0 = 32 ms^{-1}
time, t = 1·2 sec

(a) If the velocity of the body after 1·2 s is $\vec{v}$ then we know, the horizontal component of the velocity,

$$v_x = v_0 \cos\theta = 32 \times \cos 30^\circ$$
$$= 32 \times 0.866 \text{ ms}^{-1} = 27.71 \text{ ms}^{-1}$$

and vertical compnent,

$$v_y = v_0 \sin\theta - gt$$
$$= 32 \times \sin 30^\circ - 9.8 \times 1.2$$
$$= 4.24 \text{ ms}^{-1}$$

$\therefore$ magnitude of the velocity, $|\vec{v}| = \sqrt{v_x^2 + v_y^2}$

$$= \sqrt{(27.71)^2 + (4.24)^2}$$
$$= 28 \text{ ms}^{-1}$$

$$\therefore \tan\theta = \frac{v_y}{v_x} = \frac{4.24}{27.71}$$

$$\therefore \theta = \tan^{-1}\frac{4.24}{27.71} = 8.698^\circ$$

From the stimulus, we get,
ejection velocity, v_0 = 32 ms^{-1}
angle of ejection, θ = 30°
distance of the wall AB, x = 85 m
height of the wall AB, y = 2 m

(b) Suppose, if the body is ejected at an angle of θ, then the body will just cross over the wall AB.

We know,

$$y = (\tan\theta)x - \frac{gx^2}{(2v_0 \cos\theta)^2}$$

$$\text{or, } 2 = \tan\theta \times 85 - \frac{9.8(85)^2}{2(32\cos\theta)^2}$$

$$\text{or, } 2 = \tan\theta \times 85 - \frac{34.573}{\cos^2\theta}$$

$$\text{or, } 2 = \tan\theta \times 85 - \sec^2 \times (34.573)$$

$$\text{or, } 2 = \tan\theta \times 85 - 34.573\,(1 + \tan^2\theta)$$

$$\text{or, } 2 = \tan\theta \times 85 - 34.573 - 34.573\tan^2\theta$$

$$\text{or, } 34.573\tan^2\theta - 85\tan\theta + 36.573 = 0$$

$$\therefore \theta = 62.24^\circ \text{ or } \theta = 29.07^\circ$$

So, the minimum reduction of the ejection angle is (30° − 29·07°) = 0·93° then the projectile will be obstructed by the wall AB.

Investigation : Determination of acceleration of the student in 100 m race and analysing it in the graph.

Objective : Determination of average speed at different times, drawing of a graph of velocity versus time and determination of average acceleration at any time.

Apparatus : Metre scale, stop watch, rope or a measuring tape, speedometer that will be fixed to individual's shoes.

Procedure :

1. Give marking with chalk powder at one end of the playground of the college (if there is no college ground, then in any other playground).
2. Put 5 markings at 20 m interval, so that 5th marking will measure 100 m.
3. You stand near the first marking and your four friends will stand with stop watch near other 4 markings .
4. As soon as the teacher blows the whistle you start running and each of your friends will start his stop watch.
5. When the runner crosses each friend he will stop his stop watch. From the speedometer velocity will be found out.
6. By dividing the distance travelled by time taken velocity for that distance can be found out.
7. In a graph paper put time (t) along X-axis and velocity (v) along Y-axis and draw the graph.
8. From the graph velocity at any time and acceleration for a particular time interval can be found out.
9. Draw the graph again. Now find out the instantaneous acceleration for any two times.
10. By walking and running with different speeds repeat the experiment.
11. In this way each student should complete the experiment.

Table of observation

No. of observation	Distance travelled (m)	Time (sec)	Velocity v (ms^{-1})	Average acceleration $= \frac{\text{Velocity}}{\text{Time}}$ ms^{-2}
1				
2				
3				
4				

Summary

Reference frame : The coordinate system in respect of which the position of a body can be determined is called reference frame.

Displacement : Displacement is a vector whose magnitude is equal to the minimum distance between the initial and the final positions of the path of motion of the body and the direction is along the initial to final position.

Average speed : Total distance travelled by a moving body divided by the time taken for the travel is called average speed.

Instantaneous speed or speed : If the time interval approaches zero, the rate of change of distance with time is called instantaneous speed or speed.

Average velocity : Time rate of change of displacement is called average velocity.

Instantaneous velocity or velocity : If the time interval approaches zero, the rate of change of displacement with time is called instantaneous velocity or velocity.

Uniform velocity : If the velocity always remains constant, then the velocity is called uniform velocity.

Average acceleration : The average acceleration of a body is its change in velocity divided by the time required for that change.

Instantaneous acceleration or acceleration : If the time interval approaches zero, the rate of change of velocity with time of a body is called the instantaneous acceleration or simply acceleration.

Uniform acceleration : If the acceleration always remains constant, then that acceleration is called uniform acceleration.

Laws of falling bodies : There are three laws of falling bodies which are shown below :

First law : In vacuum, all the freely falling bodies starting from rest traverse equal distance in equal interval of time.

Second law : Starting from rest, the velocity of a freely falling body is proportional to the time taken to fall.

Third law : Starting from rest, the distance traversed by a freely falling body is proportional to the square of the time required.

Projectile : When an object is thrown obliquely into space, it is called a projectile and its motion is called projectile motion.

Time of flight : Time taken from the point of projection and return to the ground is called time of flight.

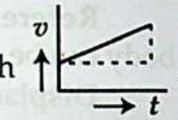
Range : The distance of the point at which it falls on a plane is called range. Or, distance between the point of projection and the end of the flight is called range.

Circular motion : If a particle or an object moves centering a point or an axis in a circular orbit then that motion is called circular motion.

Uniform circular motion : If a particle travels around a circular path at constant speed then that motion is called uniform circular motion.

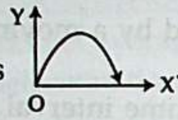
Centripetal acceleration : An object moving in a circular path with a constant speed has an acceleration called centripetal acceleration. It is directed towards the centre of the circle.

Summary of the relevant topics for the answer of multiple choice questions

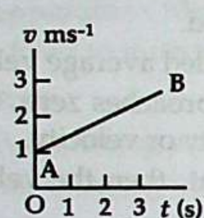
1. The equation $v = u + at$ can be expressed by the adjoining $v - t$ graph 

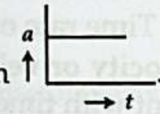
.

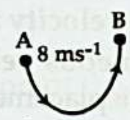
The slope of the graph = acceleration.

2. Graph for a projectile is 

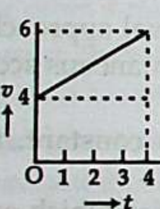
. Maximum height of the projectile, $H = \frac{v_0^2 \sin^2\theta}{2g}$.

3. 

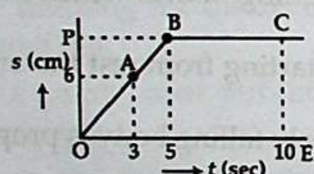
In the light of the graph of non-uniform velocity acceleration can be expressed by this graph 

. 

if 8 ms^{-1} and 10 ms^{-1} are velocities of A and B, then change of velocity is 18 ms^{-1}. $s = v_0 t + \frac{1}{2}at^2$, $s_{th} = v_0 + \frac{1}{2}a(2t - 1)$, $h = v_0 - \frac{1}{2}g(2t - 1)$ none is applicable for a body moving with variable acceleration.

4. 

From the graph magnitude of acceleration is found, from slope $= \frac{2}{4} = 0{\cdot}05$ ms^{-2} and its equation $v = v_0 + at$.

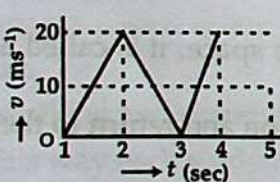
5. 

In the adjoining graph

(a) velocity at point A $= \frac{6}{3} = 2$ cms^{-1}

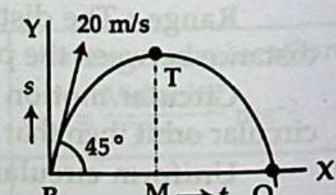
(b) line BC represents steady state of the body.

(c) distance travelled in 10 s is equal to the area of OPBCE.

6. 

(a) According to the graph distance travelled from $t = 0$ to $t = 5$ s by the body will be $s = vt = 10 \times 5 = 50$ m

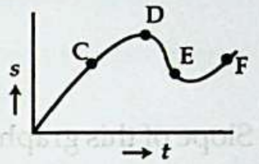
(b) velocity of the body for $t = 0$ to $t = 5$ s $= 10$ ms^{-1}

7. Ignoring the resistance of air, a stone is thrown obliquely from point P according to the adjacent graph. The maximum point of the trajectory is T and the stone reaches the point O just before touching the ground. 

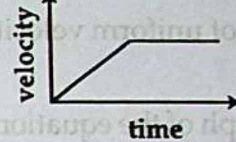
(a) Maximum horizontal range of the stone will be 40·8 m. PM = 20·4 m.

(b) Horizontal component of velocity of the stone at point T is zero. Time taken to reach the point O is 2·885 sec.

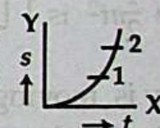
8. (a) Instantaneous velocity of the particle at point E in the following graph will be negative.



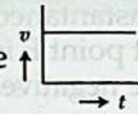
(b) (i) in this graph initial velocity is zero.
(ii) the body will never stop.



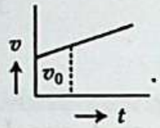
9. Angle between the velocity and acceleration at the maximum position of a projectile is $\frac{\pi}{2}$.
10. If the range is minimum ejection angle will be 0°. Trajectory of a projectile at the maximum height is one dimensional.
11. If the ejection angle is $\tan^{-1}(4)$, then the value of maximum height and horizontal range will be mutually equal.
12. Starting from rest, velocity of a freely falling body is proportional to the time taken to fall.
13. A projectile is thrown at angle of 45° with kinetic energy E. At the highest point potential energy will be, $\frac{E}{2}$.
14. Trajectory of an ejected body along the horizontal is a parabola. As the acceleration along the horizontal is zero hence component of velocity remains constant.
15. If it moves with uniform acceleration the graph in the mentioned figure will be a parabola. At the highest position velocity is zero.



16. Angular velocity of second-hand of a clock is $\frac{\pi}{30}$ rad s^{-1} and angular velocity of hour-hand is $\frac{\pi}{6}$ rad s^{-1}.
17. For maximum range a projectile is to be ejected at an angle of 45°.
18. Horizontal component of acceleration at any point in the path of a projectile is zero.
19. Equation of displacement of a moving body, $x = (4t^2 + 3t)$ m. After 2 sec velocity of the body is 19 ms^{-1}.
20. At the maximum point of the trajectory of a projectile velocity and acceleration are perpendicular to each other and angle between them is $\frac{\pi}{2}$. Velocity of horizontal component is zero.
21. If a body falls in t second after ascending height h, then after time $\frac{t}{2}$ s the body will be at a height of $\frac{3h}{4}$ from the ground.

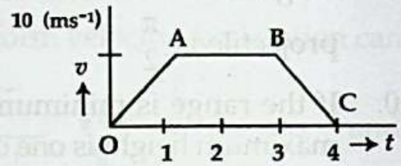
22. If the velocity of a body is constant but not zero, in that case 

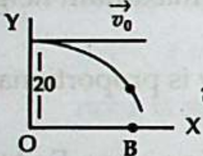
the graph will be of uniform velocity.

23. The graph of the equation $v = v_0 + at$ is 

. Slope of this graph is acceleration.

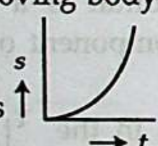
24. Trajectory of a projectile is a parabola. A particle is rotating in a circular path. The direction of its acceleration is along the centre. In the graph, in going from O to A acceleration is 10 ms^{-2}.

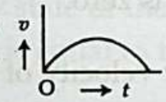


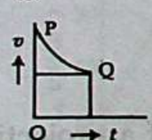
$x^2 = 80\,y$ and 

$v_0 = 20\ ms^{-1}$ obtained from B(x, y)

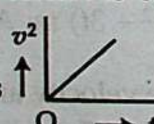
Hints : $2y\dfrac{v_0^2}{g} = x^2 = 80\,y$

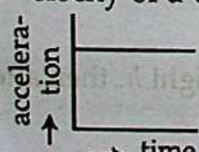
5. In case of a moving body starting from rest and at time t, graph of the equation $s = ut + \frac{1}{2}at^2$ is 

26. A body is moving from rest with non-uniform acceleration in such a way as the magnitude of its maximum acceleration slowly decreases to zero. Adjoining graph of velocity versus time indicates it. 



equation of v in PQ curve, $v = at$

27. Graph of v^2 versus t is of a particle started from rest and moving with uniform acceleration is 

28. Uniform increase of velocity of a car moving in straight path can be expressed by the adjoining graph 

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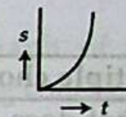
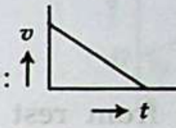
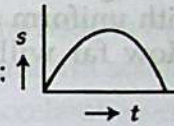
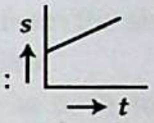
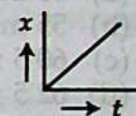
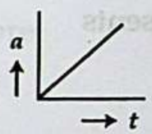
29. When $t = 0$, a body starting from rest moves with constant acceleration. The adjoining graph indicates the change of displacement with time correctly. 

30. A body is thrown with velocity v vertically upward and after time t it comes back to the ground.

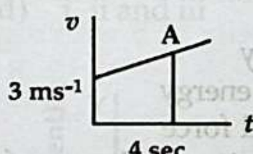
(a) In this case graph of velocity versus time is : 

(b) In this case graph of displacement versus time is : 

31. In the graph of x versus t, uniform velocity graph is :  or, 

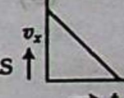
32. If the equation $x = \frac{1}{3}t^2 + 3t$ represents the displacement of a body, then the graph of acceleration versus time will be  .

33. Motion of a projectile at maximum height is one dimensional. Acceleration of a projectile along X-axis is $a_x = 0$, acceleration along Y-axis, $a_y = -g$. As there is no acceleration along the horizontal direction, the velocity along horizontal direction is constant.

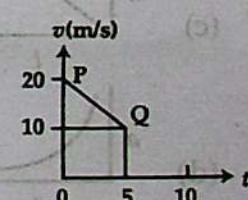
34. 

(a) slope of the graph indicates acceleration, $a = \frac{v}{t}$.
(b) velocity at point A, $v = u + at = 3 + 2 \times 4 = 11\ ms^{-1}$

35. For the same ejection velocity and for same range if one of the ejection angle is 30°, then the other angle will be 60°.
Explanation : if $\theta = 90° - \alpha$ then R becomes same.
$\therefore \quad \theta = 90° - 30° = 60°$

36. In case of motion of a projectile graph of horizontal velocity versus time is 

37. If $x = 12t - 1.2\,t^2$ then at 3 sec velocity will be $4.8\ ms^{-1}$ and acceleration $-2.4\ ms^{-2}$.



38. For PQ graph $v = at$ is applicable.